STRUCTURAL DESIGN NOTES
TOPIC C
PRESSURE VESSEL STRESS ANALYSIS

1. INTRODUCTION

These notes supplement class lectures on "thin shell" pressure vessel stress analysis.

The use of the simplified thin shell methods are illustrated by application to a pressure vessel that has many of the geometric and operational features of a pressurized water reactor (PWR) reactor vessel. More detailed analyses (e.g., by using a finite element computer code applied to a more realistic geometry) would undoubtedly be used in a final design. However, the simplified techniques can be used to give approximate answers (and answers that are easily understood) for many actual stresses of interest.

The reactor vessel is only one of a large number of nuclear reactor plant components for which stress analyses must be performed. Hence, in one sense, the analyses here are being used to represent many other calculations. But the reactor vessel is also a component of very special significance. That is, the reactor vessel is, for most purposes, considered to be designed, constructed, and operated so that a catastrophic (rapid or brittle) failure is incredible. Calculations analogous to those of this note and corresponding detailed analyses are used to support statements of incredibility.

2. DESCRIPTION OF REPRESENTATIVE VESSEL

The representative reactor vessel shown in Fig C-1 is chosen as a specific example. The overall height of the vessel (including both closure heads) is 13.3 m and the inside diameter is 4.4 m. Subregions[1] of interest, with some approximate dimensions, are:

- the <u>lower head</u> region (approximately a hemispherical shell of thickness 120 mm);

- the <u>beltline</u> region (a cylindrical shell of thickness 220 mm);

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[1] These subregions are mostly constructed of a ductile low-carbon steel (such as type SA533B). However, an austenitic stainless steel (such as SS304) covers all portions of the low-carbon steel that are adjacent to the coolant. The purpose of the stainless steel clad is corrosion protection. It has about a 3 mm minimum thickness and about a 5 mm average thickness.

- the <u>nozzle shell course</u> region (a thick cylindrical shell of thickness 380 mm with large penetrations for two hot leg pipes (1.1 m inside diameter) and four cold leg pipes (0.8 m inside diameter);

- the <u>closure flange</u> (a heavy ring that is welded to the closure head and contains holes through which closure studs pass); and

- the <u>closure head</u> (approximately a hemispherical shell with penetrations for instrumentation and for control element assemblies).

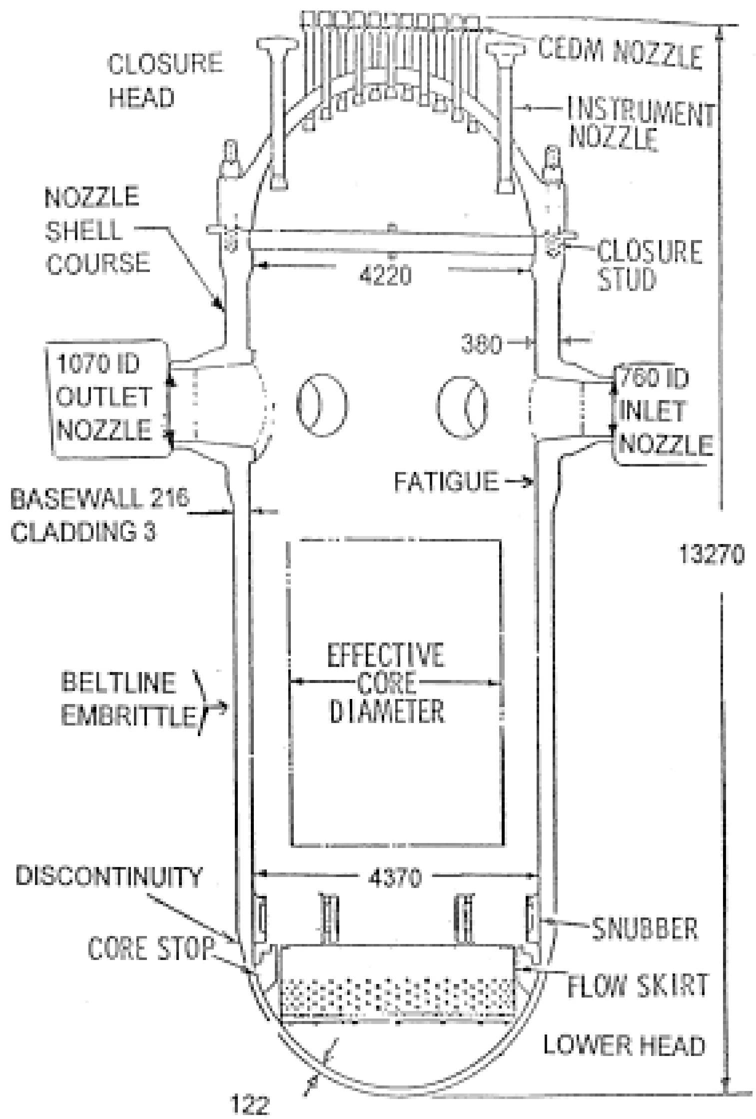

Figure C-1: A Representative PWR Reactor Vessel (adapted from Ref 1) Dimensions in millimeters.

Potential structural limits are listed as follows, as are locations of major severity for each:

<u>Structural Limit</u>	<u>Locations of Major Severity</u>
Pressure stress	beltline cylinder, lower head hemisphere, and closure head hemisphere (away from joints with adjacent regions in beltline cylinder)
Thermal stress	in the thick portion of the nozzle shell course region adjacent to a main coolant pipe penetration

Discontinuity stress	in vicinity of joints between beltline cylinder and lower head hemisphere, between beltline cylinder and nozzle shell course region cylinder, and between closure head and the closure flange
Radiation embrittlement	in the beltline cylinder adjacent to the axial center of the reactor core

3. PRESSURE STRESS

The pressure stress limits may be discussed by considering a vessel that is constructed of a thin cylindrical shell of length L that is capped by a hemisphere at either end. The mean radius of the cylinder (and the caps) is denoted by R. The cylinder has a uniform thickness equal to t_c; each cap has a uniform thickness equal to t_s. The vessel is subjected to an internal pressure (p) and a zero external pressure. No other external forces act. The vessel walls are at a uniform temperature and are constructed of a single material.

3.1 Long Cylinder

In a region of the cylinder that is far from the ends, three normal stresses (σ_r, σ_θ, and σ_z) may be calculated to characterize the thin shell stress state. These stress components are, respectively, stresses in the radial, hoop, and axial directions. The stresses σ_θ, and σ_z are found from equations of static equilibrium. The stress σ_r is obtained by averaging the pressures on the inner and outer wall. Therefore:

$$\sigma_\theta = \left(\frac{p\,R}{t_c}\right) ; \tag{1}$$

$$\sigma_z = \left(\frac{p\,R}{2\,t_c}\right) ; \text{ and} \tag{2}$$

$$\sigma_r = -\left(\frac{1}{2}p\right) . \tag{3}$$

An elastic calculation of strain in the hoop direction (θ direction) can be converted to w_c, the radially outward displacement of the center surface of the cylinder.

$$w_c = \left(\frac{p\,R^2}{2E\,t_c}\right)\left(2 - \nu + \nu\left(\frac{t_c}{R}\right)\right) ; \tag{4}$$

where: the first term on the right hand side of Eq 4 gives the σ_θ contribution to w_c; the second term, the σ_z contribution; and the last term, the σ_r contribution.

3.2 Sphere

In any portion of the hemispheres which act to give displacements and stresses that are the same as those in a full sphere,[2] the following analogous stress and displacement equations apply (the subscripts $\phi 1$ and $\phi 2$ refer to two orthogonal directions within the shell center surface):

$$\sigma_{\phi 1} = \sigma_{\phi 2} = \left(\frac{p\,R}{2\,t_s}\right) \quad ; \tag{5}$$

$$\sigma_r \;=\; -\left(\frac{1}{2}\,p\right) \quad ; \text{ and} \tag{6}$$

$$w_s \;=\; \left(\frac{p\,R^2}{2E\,t_s}\right)\left(1 - \nu + \nu\left(\frac{t_s}{R}\right)\right) \quad . \tag{7}$$

4. THERMAL STRESSES

Thermal stress calculations may be illustrated by considering a cylinder that is subjected to a known temperature distribution. The distribution is a function only of x (the distance measured radially outward from a position (x = 0) at the shell center surface). Thus, the inner surface of the shell is located at $x = -\frac{1}{2} t_c$ and the outer surface is at $x = +\frac{1}{2} t_c$. The temperature distribution (T(x)) is converted to a "thermal strain" distribution ($\varepsilon_T(x)$) by using an integrated α_T (the coefficient of linear thermal expansion), as follows:

$$\varepsilon_T = \int_{T_R}^{T} \alpha_T \, dT \quad ; \tag{8}$$

where T_R is a convenient reference temperature (e.g., 20°C); and where this integral provides ε_T as a function of T. The thermal stresses may be expressed in terms of the spatially average thermal strain ($\bar{\varepsilon}_T$) as follows:

$$\bar{\varepsilon}_T = \frac{1}{t_c} \int_{-\frac{1}{2}t_c}^{+\frac{1}{2}t_c} \left(\varepsilon_T\right) dx \quad ; \tag{9}$$

where, by evaluating elastic stress-strain relations, by requiring that axial strains are uniform (at a position far from the ends of the cylinder); and by invoking zero internal pressure and zero axial force:

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[2] These portions of the hemispheres would have no shell moments and no shell shear forces.

$$\varepsilon_z = \varepsilon_\theta = \bar{\varepsilon}_T \quad \text{; and} \tag{10}$$

$$\sigma_z = \sigma_\theta = \left(\frac{E}{1-\nu}\right)\left(\bar{\varepsilon}_T - \varepsilon_T\right) \quad ; \tag{11}$$

where ε_T indicates the local value of thermal strain at a specified x-position.

5. BENDING (CURVATURE) OF A CYLINDRICAL SHELL

In §3 and §4, the radial displacement (w) is uniform. Now consider a portion of the cylindrical shell in which the shell may have an axial slope (non-zero value of θ = dw/dz) and an axial curvature (non-zero value of d^2w/dz^2). The existence of such portions of the shell must be caused by shell moments and shell shears. Those moments and shears would typically develop near the ends.

The developments that follow are based on slopes and curvatures which may vary in the axial direction but have no changes in the hoop direction.

5.1 Strain Relations

The shell curvature results in a "bending strain" at any cross-section given by ε_{bz}:

$$\varepsilon_{bz} = -\frac{t_c}{2}\frac{d^2w}{dz^2} \quad ; \tag{12}$$

The corresponding tensile strain variation through the shell thickness is taken to be linear (analogous to a statement of beam theory that "plane sections that are originally normal to the beam axis remain plane and normal to the beam axis in the deflected condition").

$$\varepsilon_z = \bar{\varepsilon}_z + \left(\frac{2x}{t_c}\right)\varepsilon_{bz} \quad . \tag{13}$$

The tensile strains in the hoop direction are uniform at each axial position:

$$\varepsilon_\theta = \bar{\varepsilon}_\theta = (w/R) \quad . \tag{14}$$

5.2 Stress Relations

The stresses at each position may be related to shell forces and shell moments (N = normal force per unit center surface length; M = moment per unit center surface length; subcripts indicate the normal direction for the face on which N or M acts) as follows:

$$\sigma_z = \left(\frac{N_z}{t_c}\right) - \left(\frac{12\,M_z}{t_c^{\,3}}\right)x \quad ; \tag{15}$$

$$\sigma_\theta = \left(\frac{N_\theta}{t_c}\right) - \left(\frac{12\,M_\theta}{t_c^{\,3}}\right)x \quad \text{; and} \tag{16}$$

$$\sigma_r = -\left(\frac{1}{2}p\right) \quad . \tag{17}$$

5.3 Shell Variable Relations

The shell shear force variable (V_z = the shear force normal to the axial direction (per unit center surface length)) may be related to other shell variables by using force and moment balances, as follows. Each equation is preceded by the name of an analogous beam theory equation.

shear[3] $$\frac{dV_z}{dz} = p - \frac{N_\theta}{R} \;; \text{ and} \tag{18}$$

moment $$\frac{dM_z}{dz} = V_z \quad . \tag{19}$$

Additional analogs of beam theory equations are as follows:

slope $$\frac{d\theta}{dz} = M_z/D \quad ; \tag{20}$$

displacement $$\frac{dw}{dz} = \theta \quad ; \tag{21}$$

where[4] $$D = \frac{E\,t_c^3}{12(1-\nu^2)} \quad . \tag{22}$$

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[3] The name "shear" implies that Eq 18 could, in principle, be integrated to obtain a shear distribution. The hoop normal force N_θ is however a linear function of w(z). w(z) must be known prior to integration. This feature identifies a "beam on elastic foundation."

The moment in the hoop direction may be considered to be induced by the moment in the axial direction since:

$$M_\theta = \nu M_z \quad . \tag{23}$$

Other equations that relate shell variables are:

$$N_\theta \;=\; \frac{E\,t_c\,w}{R} \;+\; \nu N_z \;-\; \frac{\nu t_c\,p}{2} \quad ; \quad \text{and} \tag{24}$$

(24)

$$N_z \;=\; \frac{pR}{2} \quad . \tag{25}$$

5.4 Differential Equation

Combine many of the above equations to obtain a differential equation for w as a function of z:

$$\frac{d^4 w}{dz^4} \;+\; 4\beta^4 w \;=\; 4\beta^4 w_p \quad ; \tag{26}$$

where $w = w_p$ is a particular solution to the differential equation and is a result identical to Eq 4:

$$w_p \;=\; \left(\frac{p\,R^2}{2E\,t_c}\right)\left(2 \,-\, \nu \,+\, \nu\left(\frac{t_c}{R}\right)\right) \quad ; \tag{27}$$

and where β is given by:

$$\beta \;=\; \left(\frac{3(1-\nu^2)}{R^2\,t_c^2}\right)^{1/4} \quad . \tag{28}$$

5.5 Solution

The general solution of Eq 26 is:

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4 The symbol D denotes “flexural rigidity” and plays a role that is similar to the product (EI) in beam theory.

$$w = \left\{ \begin{array}{c} w_p + \\ \exp(-\beta z)\left[c_1 \cos(\beta z) + c_2 \sin(\beta z)\right] + \\ \exp(-\beta(L-z))\left[c_3 \cos(\beta(L-z)) + c_4 \sin(\beta(L-z))\right] \end{array} \right\} \quad ; \tag{29}$$

where the first line on the right hand side is the particular solution of Eq 27; the second line has a decaying envelope that dimishes as z increases from zero; and the final line has a decaying envelope that dimishes as z decreases from z = L. The envelopes reach a small value (exp (-3) = 0.050) as [(distance from end) = (3/β)] so that disturbances caused by end moments or by end shears are not felt at large distances.

6. DISCONTINUITY STRESSES

6.1 End of Long Cylinder

If occurrences near z = L decay in the manner indicated above, then displacements near z = 0 can be considered to depend only on the shear & moment at z = 0. That is, if the subscript "o" denotes occurrences at z = 0, then the end radial displacement and the end slope are:

$$w_o = w_{pc} + \left(\frac{1}{2\beta^3 D}\right) V_o + \left(\frac{1}{2\beta^2 D}\right) M_o \quad \text{; and} \tag{30}$$

$$\theta_o = -\left(\frac{1}{2\beta^2 D}\right) V_o - \left(\frac{1}{\beta D}\right) M_o \quad . \tag{31}$$

6.2 Edge of Hemisphere

A similar, but more involved, equation development leads to the following equations for the hemisphere that is joined to the cylinder at z = 0:

$$w_o = w_{ps} - \left(\frac{2R\beta}{E t_s}\right) V_o + \left(\frac{2\beta^2}{E t_s}\right) M_o \quad \text{; and} \tag{32}$$

$$\theta_o = -\left(\frac{2\beta^2}{E t_s}\right) V_o + \left(\frac{4\beta^3}{R E t_s}\right) M_o \quad \text{; where} \tag{33}$$

$$\lambda = \beta_s R \quad ; \text{ and} \tag{34}$$

$$\beta_s = \left[\frac{3(1-\nu^2)}{R^2 \, t_s^2} \right]^{1/4} \quad . \tag{35}$$

6.3 Combination

We now have four linear equations (Eqs. 30-33) in four unknowns (w_o, θ_o, M_o, and V_o) and therefore can solve for conditions at the joint (z = 0) that give continuity. Subsequently, corresponding values of c_1 and $c_2$1 (based on Eq. 29; $c_3 = 0$; and $c_4 = 0$) can be found and used to determine conditions in the cylinder for other values of z.

7. RELATED INFORMATION

Related information on pressure vessel stress analyses can be found as follows:

- thin shell pressure stresses (Ref. 2, pp. 27-32; Ref. 3, pp. 33-45);
- thick shell pressure stresses (Ref. 2, pp. 32-37; Ref. 3, pp. 56-64);
- thermal stresses (Ref. 2, pp. 37-44; Ref. 3, pp. 74-88);
- discontinuity stresses (Ref. 3, pp. 159-185); and
- combination information for a cylindrical shell (Ref. 4) and for a spherical shell (Ref. 5).

Be aware that in this note and in the references, a variety of approximations and notations are used. For example, the thin shell approach adopted herein does take some account for the shell "squeezing" caused by pressure action in the radial direction. Other references adopt an assumption that the effect is negligible. The definition of the radius R provides a second example. It is used herein as the radius of the shell center surface but is used elsewhere as the shell inside radius.

Other reference-to-reference differences exist; therefore each reference should be used with care. However, each reference provides a unique viewpoint and derivation approach. It can be used as a useful supplement to the information of class lectures and this note.

REFERENCES

1. "Reactor Vessel Information," MIT Nuclear Engineering Department Notes L.7, from Pilgrim station Unit 2, Preliminary Safety Analysis Report (PSAR).

2. L. Wolf, M.S. Kazimi, and N.E. Todreas, "Introduction to Structural Mechanics," MIT Nuclear Engineering Notes L.4, revision of Fall 1995.

3. J. F. Harvey, "Theory and Design of Pressure Vessels," Van Nostrand Reinhold Co., New York, 1985.

4. "Article A-2000, Analysis of Cylindrical Shells," in Section III, Rules for Construction of Nuclear Vessels, ASME Boiler and Pressure Vessel Code," pp. 547-549, ASME, New York, 1968 Edition.

5. "Article A-3000, Analysis of Spherical Shells," in Section III, Rules for Construction of Nuclear Vessels, ASME Boiler and Pressure Vessel Code," pp. 553-556, ASME, New York, 1968 Edition.

APPENDIX C1
Stresses Near Axial Center of Long Cylinder
Comparison of Alternate Approximations
J.E. Meyer
revision of July 1996

PRESSURE STRESSES (at r = a)

	(STRESS at r = a) / (PRESSURE)					
	Equations for Stress at r = a			(σ / p) for (b / a) = 1.1		
	radial	hoop	axial	radial	hoop	axial
Exact, Thick Shell	- p	1b	1c	-1.00	10.52	4.76
Thin Shell for Curvatures	- (p / 2)	2b	2c	-0.50	10.50	5.25
Ring Finite Element	3a	3b	3c = 1c	-0.48	10.00	4.76

$$\sigma_\theta = \left(\frac{b^2 + a^2}{b^2 - a^2}\right) p \quad (1b) \qquad \sigma_z = \left(\frac{a^2}{b^2 - a^2}\right) p \quad (1c) \qquad \sigma_\theta = \frac{p\,R}{t} \quad (2b)$$

$$\sigma_z = \frac{p\,R}{2\,t} \quad (2c) \qquad \sigma_r = -\left(\frac{a}{b + a}\right) p \quad (3a) \qquad \sigma_\theta = \left(\frac{a}{b - a}\right) p \quad (3b)$$

Symbols are: p = inside pressure; zero = outside pressure; σ = stress; (r, θ, z) = (radial, hoop, axial) coordinate directions; r = b = radius of shell outside surface; r = a = radius of shell inside surface; r = R = 0.5 (b + a) = radius of shell central surface; t = b - a = shell thickness.

THERMAL STRESSES (at r = a)

radial direction	hoop direction	axial direction
$\sigma_r = 0$	$\sigma_\theta = \frac{E}{1-\nu}\left(\bar{\varepsilon}_T - \varepsilon_{Ta}\right)$	$\sigma_z = \frac{E}{1-\nu}\left(\bar{\varepsilon}_T - \varepsilon_{Ta}\right)$

Symbols are: E = Young's Modulus; ν = Poisson's Ratio; $\bar{\varepsilon}_T$ = volume averaged thermal strain; ε_{Ta} = thermal strain at r = a.

<table>
<tr><th rowspan="2">Category</th><th colspan="3">Primary</th><th rowspan="2">Second-ary</th><th rowspan="2">Peak</th></tr>
<tr><th>General Membr</th><th>Local Membr</th><th>Bending</th></tr>
<tr><td>Symbol</td><td>P_m</td><td>P_L</td><td>P_b</td><td>Q</td><td>F</td></tr>
<tr><td>Normal Force</td><td>X</td><td>X</td><td></td><td></td><td></td></tr>
<tr><td>Moment</td><td></td><td></td><td>X</td><td></td><td></td></tr>
<tr><td>Includes Discontinuities</td><td></td><td>X</td><td></td><td>X</td><td>X</td></tr>
<tr><td>Includes Stress Concentrations</td><td></td><td></td><td></td><td></td><td>X</td></tr>
<tr><td>Caused by Mechanical Loads</td><td>X</td><td>X</td><td>X</td><td>X</td><td>X</td></tr>
<tr><td>Caused by Thermal Stresses</td><td></td><td></td><td></td><td>X</td><td>X</td></tr>
<tr><td>Failure Mode</td><td>Tensile Limit Load</td><td>Strain Control Shake-down)</td><td>Bending Limit Load</td><td>Strain Control (Shake-down)</td><td>Fatigue</td></tr>
<tr><td>Limit from Concepts</td><td>S_y</td><td>$2\ S_y$</td><td>$1.5\ S_y$</td><td>$2\ S_y$</td><td>S-N Curves</td></tr>
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<tr><td rowspan="4">Inelastic Analysis (Elevated Temperature)</td><td>□=1%</td><td colspan="4"></td></tr>
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<tr><td colspan="4"></td><td>$U_{FT} < D$</td></tr>
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