Course 22.314 "Structural Mechanics: in Nuclear Power Technology"

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THERMOELASTIC STRESSES IN A HOLLOW CYLINDER WITH THIN-WALLED INTERNAL CLADDING FOR AXISYMMETRIC PLANE STRAIN CONDITIONS

1. Introduction

Vessels and pipes are being clad or coated by corrosion resistant layers in order to prevent direct contact between the basic vessel or pipe material and the contained fluid. If the compound structure is also to be subjected to severe temperature conditions, the design should aim to minimize the effect of thermal stresses.

The following analysis is concerned with the simple case of thermoelastic stresses under axisymmetric plane strain conditions in a thick-walled long hollow cylinder with thin-walled internal cladding of thickness t (Fig. 1), where the cladding is assumed to have been inserted or bonded to the hollow cylinder under conditions not imposing prestressing to either member. The structure is subjected to a polar-symmetric temperature field $T(r)$.

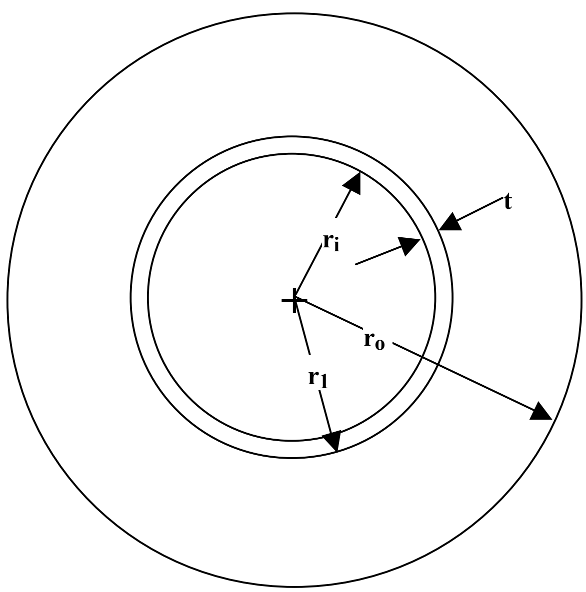
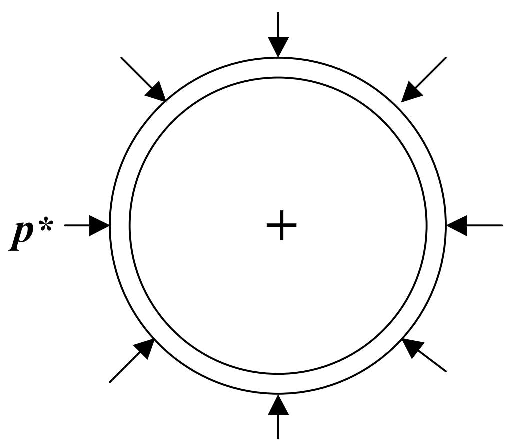


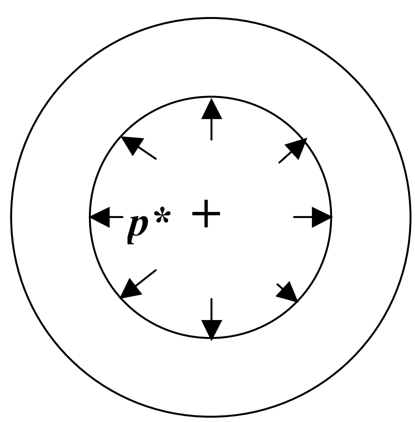
Fig. 1. The clad tube

The thermoelastic materials properties of the hollow cylinder and the cladding. i.e. the coefficient of thermal expansion α, the modulus of elasticity E, and the Poisson's ratio ν, are considered to be constant (independent of temperature). The material properties of the hollow cylinder are distinguished from those of the cladding by the use of the subscripts h for the hollow cylinder and c for the cladding. The inner surface of the cladding shell is defined by the radius r_i, the contact surface is defined by the radius r_1, and the outer surface of the hollow cylinder is defined by the radius r_o. It is assumed that the ratio $t/r_o << 1$.

In determining the thermoelastic stress-strain relations of the composite body, the hollow cylinder and the cladding are first considered as hypothetically separate free bodies (Fig. 2). Under the influence of temperature gradients and/or due to differing thermoelastic properties these hypothetically separated bodies experience different thermoelastic derformations.



The Cladding



The Basic Tube

Fig. 2 The free bodies.

However, since there is assumed to be ideal bonding between hollow cylinder and cladding, the deformations of both components of the composite body must be compatible at the contact surfaces, i.e.

$$(u_c)_{r=r_1} = (u_h)_{r=r_1} \; and \; (\varepsilon_{z,h})_{r=r_1} = (\varepsilon_{z,h})_{r=r_1} \qquad \text{(I-1,2)}$$

The stress-displacement field of the composite cylinder can be evaluated on the basis of these compatibility conditions. The equations contain two undetermined terms, namely the radial pressure p^* acting at the contact interface r_1, and the constant axial strain ε_z.

If the free radial thermal expansion of the cladding is larger than the free radial thermal expansion of the hollow cylinder internal surface, the contact pressure p^* is a compressive stress.

In the following study it is assumed that $\alpha_c > \alpha_h$, implying that $p^* < 0$ throughout.

2. Free Body Equations

2.1 Thick-Walled Hollow Cylinder

The following conditions are imposed on the thick-walled hollow cylinder:

$$(\sigma_r)_{r=r_1} = -p^*, \qquad (\sigma_r)_{r=r_o} = 0, \qquad T = T(r). \qquad (1)$$

Using these conditions and introducing an average temperature $\hat{T}$ according to Eq. [M-11/(34)] for the region $r_1 \le r \le r_o$, into the general thermoelastic relations for the axisymmetric plane (polarsymmetric) state of plane strain, Eqs. [M-11/(16)-(19)], leads to the following equations:

$$\frac{E_h u_h}{1+\nu_h} = \frac{E_h \alpha_h}{(1-\nu_h) r} \int_{r_1}^{r} rT(r)\,dr + \frac{E_h \alpha_h}{2(1-\nu_h)} \left[(1-2\nu_h) r + \frac{r_1^2}{r} \right] \hat{T}$$
$$+ \frac{p^* r_1^2}{r_o^2 - r_1^2} \left[(1-2\nu_h) r + \frac{r_o^2}{r} \right] - \frac{E_h \nu_h \varepsilon_z r}{1+\nu_h}; \qquad (2)$$

$$\sigma_r = -\frac{E_h \alpha_h}{(1-\nu_h) r^2} \int_{r_1}^{r} rT(r)\,dr + \frac{E_h \alpha_h}{2(1-\nu_h)} \left(1 - \frac{r_1^2}{r^2} \right) \hat{T} + \frac{p^* r_1^2}{r_o^2 - r_1^2} \left(1 - \frac{r_o^2}{r^2} \right) \qquad (3)$$

$$\sigma_\theta = \frac{E_h \alpha_h}{(1-\nu_h) r^2} \int_{r_1}^{r} rT(r)\,dr - \frac{E_h \alpha_h}{1-\nu_h} T(r) + \frac{E_h \alpha_h}{2(1-\nu_h)} \left(1 + \frac{r_1^2}{r^2} \right) \hat{T}$$
$$+ \frac{p^* r_1^2}{r_o^2 - r_1^2} \left(1 + \frac{r_o^2}{r^2} \right) \qquad (4)$$

$$\square_z = E_h \square_z \square \frac{E_h \square_h}{1 \square \square_h} T(r) + \frac{\square_h E_h \square_h}{1 \square \square_h} \hat{T} + \frac{2\square_h \, p \square r_1^2}{r_o^2 \square r_1^2} . \tag{5}$$

2.2 Thin-Walled Cladding

The following conditions are imposed on the thin-walled cladding:

$$\left(\square_r\right)_{r=r_1} = 0, \quad \left(\square_r\right)_{r=r_1} = \square p \square, \quad T \square T_c = T_o = const. \tag{6}$$

whereby Eqs. [M-11/(16)-(19)] reduce to:

$$\frac{E_c u_c}{(1+\square_c) r} = E_c \square_c T_o \square \frac{p\square}{\frac{t}{r}\square 2 \square \frac{t}{r_1}\square} \square 1 \square 2\square_c + \square \frac{r_1 \square t}{r} \square^2 \square \square \frac{E_c \square_c \square_z}{1+\square_c} \; ; \tag{7}$$

$$\square_r = \square \frac{p\square}{\frac{t}{r_1}\square 2 \square \frac{t}{r_1}\square} \square 1 \square \square \frac{r_1 \square t}{r} \square^2 \square , \tag{8}$$

$$\square_\square = \square \frac{p\square}{\frac{t}{r_1}\square 2 \square \frac{t}{r_1}\square} \square 1 + \square \frac{r_1 \square t}{r} \square^2 \square , \tag{9}$$

$$\square_z = E_c \square_z \square E_c \square_c T_o \square \frac{2\square_c \, p\square}{\frac{t}{r_1}\square 2 \square \frac{t}{r_1}\square} . \tag{10}$$

In case $\left(t/r_1\right) << 1$ there follows from $\square 1 \square \frac{t}{r_1} \square \square \frac{r}{r_1} \square 1$, that $r \square r_1$. In this case the solution reduces to the still further simplified expressions:

$$\frac{E_c u_c}{(1 \square \square_c) r_1} = E_c \square_c T_o \square p \square \frac{r_1}{t} \left(1 \square \square_c\right) \square \frac{E_c \square_c \square_z}{1+\square_c} \; ; \tag{11}$$

$$\square_r = 0 , \quad \square_\square = \square p \square \frac{r_1}{t} , \qquad \square z = E_c \square_c \square \square_c E_c T_o \square \square_c \, p \square \frac{r_1}{t} . \tag{12a,b,c}$$

3. Compatibility Conditions of the Compound Body

Employing the condition of compatibility of radial displacements for the composite body,

$$\left(u_c\right)_{r=r_1} = \left(u_h\right)_{r=r_1} \tag{13}$$

there follows from Eqs. (2) and (11):

$$(1+\nu_c)\,H + (\nu_c - \nu_h)\,\alpha_h\,\hat{T} = \varepsilon_z(\nu_c - \nu_h) + p^*\left[\frac{(1-\nu_h)\left(r_o^2/r_1^2 + 1 - 2\nu_h\right)}{E_h\left(r_o^2/r_1^2 - 1\right)} + \frac{(1+\nu_c)\,(1-\nu_c)}{E_c}\,\frac{r_1}{t}\right] \tag{14}$$

where $H = \alpha_c T_o - \alpha_h \hat{T}$. (14a)

Since with a thin-walled cladding the ratio r_1/t is so large that the relevant term in Eq. (14) is by far the dominant one in the coefficient expression for p^*, Eq. (14) can be simplified to become

$$(1+\nu_c)H + (\nu_c - \nu_h)\alpha_h\hat{T} = \varepsilon_z(\nu_c - \nu_h) + \frac{p^*(1+\nu_c)(1-\nu_c)}{E_c}\,\frac{r_1}{t}\ . \tag{15}$$

For free cylinder ends without axial forces applied to the ends the following condition holds

$$\int_{r_1-t}^{r_1} \sigma_z\, rdr + \int_{r_1}^{r_o} \sigma_z\, rdz = 0\ , \tag{16}$$

where for $t/r_1 << 1$ the first integral is approximately equal to $\left(\sigma_z\, r_1\, t\right)$.

Utilizing Eqs. (5) and (12c) the condition Eq. (16) takes the following form:

$$E_c\,\frac{t}{r_1}H + \left[E_c\,\frac{t}{r_1} + \frac{E_h}{2}\left(\frac{r_o^2}{r_1^2} - 1\right)\right]\alpha_h\,\hat{T} = p^*(\nu_h - \nu_c) + \varepsilon_z\left[E_c\,\frac{t}{r_1} + \frac{E_h}{2}\left(\frac{r_o^2}{r_1^2} - 1\right)\right]. \tag{17}$$

For determining ε_z, Eqs. (15) and (17) are simultaneously solved for p^*:

$$p^* = \frac{E_c\,t}{r_1}\left[\frac{1}{1-\nu_c}H + \frac{\nu_c - \nu_h}{(1+\nu_c)(1-\nu_c)}\left(\alpha_c\,T_o - \varepsilon_z\right)\right],$$

$$p^* = \frac{1}{\nu_h - \nu_c}\left[\frac{E_c\,t}{r_1}H + \left[E_c\,\frac{t}{r_1} + \frac{E_h}{2}\left(\frac{r_o^2}{r_1^2} - 1\right)\right]\left(\alpha_h\,\hat{T} - \varepsilon_z\right)\right]\ ; \tag{17 a, b}$$

these equations are subtracted from each other in order to eliminate the unknown constant pressure $p*$ leading to the expression

$$\frac{E_h r_1(1-\nu_c)}{2E_c t}\left[\frac{r_o^r}{r_1^2}-1\right]\left[1+\frac{2E_c\left(1-2\nu_c\nu_h+\nu_h^2\right)}{E_h\left(1-\nu_c^2\right)\left(r_o^2/r_1^2-1\right)}\frac{t}{r_1}\right.$$

$$\left.E_c\frac{t}{r_1}+\left(\nu_c-\nu_h\right)^2\right]\left(\varepsilon_z-\alpha_h\hat{T}\right)-\left(1-\nu_h\right)H=0 \qquad (18)$$

which, for a thin-walled cladding, $t/r_1 << 1$, reduces to:

$$\varepsilon_z=\alpha_h\hat{T}+\frac{2\left(1-\nu_h\right)E_c H}{\left(1-\nu_c\right)E_h\left(r_o^2/r_1^2-1\right)}\frac{t}{r_1} \qquad (19)$$

The other unknown $p*$ can be determined by substituting Eq. (19) into Eq. (14); (substitution of the approximative solution for ε_z under the condition $t/r_1 << 1$ into the not simplified Eq. (14), of course, implies a small inaccuracy). Using a parameter β_2, as utilized by B.W. Shaffer, the following expression is obtained:

$$p*\beta_2=\frac{E_c H}{1-\nu_c}\frac{t}{r_1}\left[1-\frac{\left(\nu_c-\nu_h\right)\left(1-\nu_h\right)E_c}{\left(1+\nu_c\right)\left(1-\nu_c\right)E_h}\frac{2}{\left(r_o/r_1\right)^2-1}\frac{t}{r_1}\right], \qquad (20)$$

where

$$\beta_2=1+\frac{1+\nu_h}{1-\nu_c^2}\frac{E_c}{E_h}\frac{\left(r_o/r_1\right)^2+1-2\nu_h}{\left(r_o/r_1\right)^2-1}\frac{t}{r_1}\ . \qquad (21)$$

For $\frac{t}{r_1} << 1$ there is $\beta_2 \approx 1$ and $p* \approx \frac{E_c H}{1-\nu_c}\frac{t}{r_1}$. (22)

Herewith the originally unknown parameters of the composite structure are determined.

4. <u>Thermoelastic Relations for the Composite Body</u>

Introducing the Eqs. (19) and (20) into the equation for the radial displacement of the hollow cylinder, Eq. (2) yields:

$$\frac{u_h}{r}=\frac{1+\nu_h}{1-\nu_h}\frac{\alpha_h}{r^2}\int_{r_1}^{r} rT(r)\,dr+\left[1-3\nu_h+\left(1+\nu_h\right)\frac{r_1^2}{r^2}\right]\left[\frac{\alpha_h\hat{T}}{2(2-\nu_h)}+\frac{E_c H\beta_1(r)}{E_h\left(r_o^2/r_1^2-1\right)\beta_2}\frac{t}{r_1}\right] \qquad (23)$$

where

$$\Box_1(r) = -\frac{\left(1-\nu_h^2\right)E_c}{\left(1-\nu_c^2\right)\left(1-\nu_c\right)E_h}\,\frac{2}{\left(r_o/r_1\right)^2-1}\left[\nu_c-2\nu_c\nu_h+\nu_h\frac{r_o^2}{r_1^2}+\left(\nu_c-\nu_h\right)\frac{r_o^2}{r^2}\right]\frac{t}{r_1} \left. -\frac{2\nu_h\left(1-\nu_h\right)}{1-\nu_c}+\frac{1+\nu_h}{1-\nu_c}\left[1-2\nu_h+\frac{r_o^2}{r^2}\right]\right\}; \quad (24)$$

$$\text{and for } \frac{t}{r_1} << 1: \Box_1(r) \approx -\frac{2\nu_h\left(1-\nu_h\right)}{1-\nu_c}+\frac{1+\nu_h}{1-\nu_c}\left[1-2\nu_h+\frac{r_o^2}{r^2}\right]. \quad (24a)$$

Substitution of Eq. (19) into Eqs. (3), (4) and (5) gives:

$$\sigma_r = \frac{E_h\alpha_h}{1-\nu_h}\left\{-\frac{1}{r^2}\int_{r_1}^{r} rT(r)\,dr+\left(1-\frac{r_1^2}{r^2}\right)\frac{\hat{T}}{2}\right\}+\frac{E_c H\left(1-r_o^2/r^2\right)}{(1-\nu_c)\left(r_o^2/r_1^2-1\right)}\frac{t}{r_1} \quad (25)$$

$$\sigma_\theta = \frac{E_h\alpha_h}{1-\nu_h}\left\{\frac{1}{r^2}\int_{r_1}^{r} rT(r)\,dr+\left(1+\frac{r_1^2}{r^2}\right)\frac{\hat{T}}{2}-T(r)\right\}+\frac{E_c H\left(1+r_o^2/r^2\right)}{(1-\nu_c)\left(r_o^2/r_1^2-1\right)}\frac{t}{r_1} \quad (26)$$

$$\sigma_z = \frac{E_h\alpha_h}{1-\nu_h}\left[\hat{T}-T(r)\right]+\frac{2E_c H}{(1-\nu_c)\left(r_o^2/r_1^2-1\right)}\frac{t}{r_1}. \quad (27)$$

At the contact interface, $r = r_1$:

$$\frac{u_h}{r_1} = \alpha_h\hat{T}+\frac{E_c H\,\Box_1\left(r_1\right)}{E_h\left(r_o^2/r_1^2-1\right)\Box_2}\,\frac{t}{r_1} \quad (28)$$

$$\left.\begin{aligned}&\sigma_r = -\frac{E_c H}{1-\nu_c}\frac{t}{r_1},\quad \sigma_\theta = \frac{E_h\alpha_h}{1-\nu_c}\left(\hat{T}-T_o\right)+\frac{E_c H\left(1+r_o^2/r^2\right)}{(1-\nu_c)\left(r_o^2/r_1^2-1\right)}\frac{t}{r_1}\\ &\sigma_z = \frac{E_c\alpha_h}{1-\nu_h}\left(\hat{T}-T_o\right)+\frac{2E_c H}{(1-\nu_c)\left(r_o^2/r_1^2-1\right)}\frac{t}{r_1}.\end{aligned}\right\} \quad (29\text{ a, b, c})$$

Introducing Eqs. (19) and (20) into the relation for the radial displacement of the cladding, Eq. (7) gives:

$$\frac{u_c}{r_1} = \alpha_h \hat{T} + \frac{E_c H \phi_1(r_1)}{E_h\left(r_o^2 / r_1^2 - 1\right)\phi_2} \frac{t}{r_1} \tag{30}$$

And substitution of Eqs. (19) and (22) into Eqs. (8) through (10) yields:

$$\sigma_r = 0,\ \sigma_\theta = -\frac{E_c H}{1-\nu_c},\ \sigma_z = -\frac{E_c H}{1-\nu_c}\left[1 - \frac{2E_c(2-\nu_h)}{E_h\left(r_o^2 / r_1^2 - 1\right)} \frac{t}{r_1}\right], \tag{31 a, b, c}$$

and for $t/r << 1$

$$\sigma_r = 0, \quad \sigma_\theta = \sigma_z = -\frac{E_c H}{1-\nu_c} = -p \cdot \frac{r_1}{t}. \tag{32}$$

5. The Elastic Limit

The thermoelastic relations derived in the previous sections are valid as long as the difference expression

$$H = \alpha_c T_o - \alpha_h \hat{T} \tag{14a}$$

is small enough so that yielding does not occur. The numerical value of H at which yielding starts may be evaluated by substituting the relations for the thermoelastic stresses into an appropriate multiaxial yield criterion, e.g. the Tresca or the von Mises yield criterion. Yielding may start either in the cladding or in the hollow cylinder, and each possibility has to be investigated.

A relevant study for the Tresca yield criterion has been done by B.W. Shaffer who also investigated the conditions of plastic flow within the cladding.

6. Reference

B.W. Shaffer: Elastic-Plastic Thermal Stress Analysis of an Internally Clad Tube. Nuclear Science and Engineering 19 (1964), p. 300-309.