Course 22.314 “Structural Mechanics in Nuclear Power Technology”

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AXISYMMETRIC ELASTIC-PLASTIC PLANE STRAIN DEFORMATION OF A HOLLOW CYLINDER OF VON MISES- MATERIAL UNDER INTERNAL PRESSURE AND THERMAL CYCLING

Problem: Thermo-elastic-plastic deformation; thermal stress ratchetting
Geometry: Hollow cylinder, plane strain F(r)
Material: Isotropic; von Mises yield condition; finite stress-strain relations
Method: Semi-analytic, iterative procedure

1. Introduction

The problem of axisymmetric elastic-plastic plane strain deformation of a long hollow cylinder of elastic-ideal plastic material under internal pressure and cyclic thermal loading has been treated by J. BREE on the basis of considerations by D.R. MILLER and utilizing the general iterative method of A. MENDELSON and S.S. MANSON.

The treatment is based on the yield criterion of von Mises and on Hencky’s finite deformation theory of plasticity. Use of the finite theory of plastic deformation implies that the compatibility equations and the equilibrium equations referred to the original configuration of the material hold in terms of the current stresses and the total strains only if the total strains are sufficiently small for their squares (and higher powers) to be neglected.

2. General Elastic-Plastic Equations

Consider a long hollow cylinder with internal radius r = a and external radius r =b in a system of cylindrical coordinates r, θ, z, in which the z-axis lies along the axis of the cylinder; the principal directions of stress and strain coincide with the directions of the coordinate axes. The hollow cylinder is assumed to be closed and subjected to an internal pressure p_i, zero external pressure, and a radial temperature distribution T(r).

The equilibrium conditions for the axisymmetric plane strain case reduce to

$$\frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\theta}{r} = 0 \tag{1}$$

and

$$\int_a^b \sigma_z r dr = p_i \frac{a^2}{2}. \tag{2}$$

The relevant compatibility equation is

$$\frac{d\varepsilon_\theta}{dr} + \frac{\varepsilon_\theta - \varepsilon_r}{r} = 0\,, \tag{3}$$

and Hencky's finite stress-strain relations are:

$$\sigma_r = \lambda\left(\varepsilon_r + \varepsilon_\theta + \varepsilon_z - 3\alpha T\right) + 2G\left(\varepsilon_r - \alpha T - \varepsilon_r^p\right), \tag{4a}$$

$$\sigma_\theta = \lambda\left(\varepsilon_r + \varepsilon_\theta + \varepsilon_z - 3\alpha T\right) + 2G\left(\varepsilon_\theta - \alpha T - \varepsilon_\theta^p\right), \tag{4b}$$

$$\sigma_z = \lambda\left(\varepsilon_r + \varepsilon_\theta + \varepsilon_z - 3\alpha T\right) + 2G\left(\varepsilon_z - \alpha T - \varepsilon_z^p\right), \tag{4c}$$

where ε_r, ε_θ, ε_z are the elastic-plastic total strains; $\varepsilon_r^p, \varepsilon_\theta^p, \varepsilon_z^p$ are the total plastic strains which satisfy the condition of incompressibility $\varepsilon_r^p + \varepsilon_\theta^p + \varepsilon_z^p = 0$; λ and G are defined by $\lambda = \dfrac{\nu E}{(1+\nu)(1-2\nu)}$, $\qquad G = \dfrac{E}{2(1+\nu)}$.

The boundary conditions are:

$$\sigma_r(r)\big|_{r=a} = -p_i\,, \qquad \sigma_r(r)\big|_{r=b} = 0\,. \tag{5a,b}$$

Combining Eqs. (1), (3), and (4) yields the relationship

$$\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r^2\varepsilon_\theta\right)\right] = \frac{\nu E}{(1+\nu)(1-2\nu)}\frac{1+\nu}{1-\nu}\alpha\frac{dT}{dr} + \frac{1-2\nu}{1-\nu}\frac{d\varepsilon_r^p}{dr} + \frac{1-2\nu}{1-\nu}\frac{\varepsilon_r^p - \varepsilon_\theta^p}{r}\,, \tag{6}$$

which by integration gives:

The compatibility condition, Eq. (3), can be written in the following form:

$$\varepsilon_r = \varepsilon_\theta + r\frac{d\varepsilon_\theta}{dr} = -\varepsilon_\theta + \frac{1}{r}\frac{d}{dr}\left(r^2\varepsilon_\theta\right), \tag{8}$$

thus

$$\varepsilon_r = -\varepsilon_\theta + \frac{1+\nu}{1-\nu}\alpha T(r) + \frac{1-2\nu}{1-\nu}\varepsilon_r^p + \frac{1-2\nu}{1-\nu}\int_a^r \frac{\varepsilon_r^p - \varepsilon_\theta^p}{r}\,dr + 2C_1\,. \tag{9}$$

Substituting Eqs. (7) and (9) into Eq. (4c) and using the axial equilibrium condition Eq. (2) renders:

$$\begin{aligned}\frac{b^2 - a^2}{2}\varepsilon_z = {} & p_i a^2\frac{(1+\nu)(1-2\nu)}{2E(1-\nu)} + \frac{1+\nu}{1-\nu}\int_a^b \alpha r T(r)\,dr - \frac{\nu}{1-\nu}\int_a^b r\left(\varepsilon_r + \varepsilon_\theta\right)dr \\ & - \frac{1-2\nu}{1-\nu}\int_a^b r\left(\varepsilon_r^p + \varepsilon_\theta^p\right)dr\,.\end{aligned} \tag{10}$$

The constants of integration, C_1 and C_2, are determined by use of the radial equilibrium condition Eq. (1) and the boundary conditions Eq. (5a,b):

$$\begin{aligned}C_1 = \frac{1}{b^2 - a^2}\Bigg[& p_i a^2\frac{1-2\nu}{E} + \frac{1-3\nu}{1-\nu}\int_a^b \alpha r T(r)\,dr + \frac{1}{2(1-\nu)}\int_a^b r\left(\varepsilon_r^p + \varepsilon_\theta^p\right)dr \\ & - \frac{b^2(1-2\nu)}{2(1-\nu)}\int_a^b \frac{\varepsilon_r^p - \varepsilon_\theta^p}{r}\,dr\Bigg],\end{aligned} \tag{11a}$$

$$\begin{aligned}C_2 = \frac{1}{b^2 - a^2}\Bigg[& p_i a^2 b^2\frac{1+\nu}{E} + a^2\frac{1+\nu}{1-\nu}\int_a^b \alpha r T(r)\,dr + a^2\frac{1-2\nu}{1-\nu}\int_a^b r\left(\varepsilon_r^p + \varepsilon_\theta^p\right)dr \\ & - \frac{a^2 b^2}{2(1-\nu)}\int_a^b \frac{\varepsilon_r^p - \varepsilon_\theta^p}{r}\,dr\Bigg].\end{aligned} \tag{11b}$$

Herewith the following equations for the strain components are obtained:

$$\varepsilon_\theta = \frac{p_i a^2}{E\left(b^2 - a^2\right)}\left[(1-2\nu)+(1+\nu)\frac{b^2}{r^2}\right]+\frac{1+\nu}{(1-\nu)r^2}\int_a^r \alpha r T(r)dr$$

$$+\frac{1}{\left(b^2-a^2\right)(1-\nu)}\left[(1-3\nu)+(1+\nu)\frac{a^2}{r^2}\right]\int_a^b \alpha r T(r)dr\,,$$

$$+\frac{1-2\nu}{2(1-\nu)}\left\{\int_a^r \frac{\varepsilon_r^p-\varepsilon_\theta^p}{r}dr+\frac{1}{r^2}\int_0^r r\left(\varepsilon_r^p+\varepsilon_\theta^p\right)dr\right\}$$

$$-\frac{1}{2\left(b^2-a^2\right)(1-\nu)}\left\{\left[(1-2\nu)+\frac{a^2}{r^2}\right]b^2\int_a^b \frac{\varepsilon_r^p-\varepsilon_\theta^p}{r}dr\right.$$

$$\left.+\left[1+(1-2\nu)\frac{a^2}{r^2}\right]\int_a^b r\left(\varepsilon_r^p+\varepsilon_\theta^p\right)dr\right\}, \tag{12}$$

$$\varepsilon_r = \frac{p_i a^2}{E\left(b^2 - a^2\right)}\left[(1-2\nu)-(1+\nu)\frac{b^2}{r^2}\right]-\frac{1+\nu}{(1-\nu)}\frac{1}{r^2}\int_a^r \alpha r T(r)dr$$

$$+\frac{1}{\left(b^2-a^2\right)(1-\nu)}\left[(1-3\nu)-(1+\nu)\frac{a^2}{r^2}\right]\int_a^b \alpha r T(r)dr+\frac{1+\nu}{1-\nu}\alpha T(r)$$

$$+\frac{1-2\nu}{1-\nu}\varepsilon_r^p+\frac{1-2\nu}{2(1-\nu)}\left\{\int_a^r \frac{\varepsilon_r^p-\varepsilon_\theta^p}{r}dr-\frac{1}{r^2}\int_0^r r\left(\varepsilon_r^p+\varepsilon_\theta^p\right)dr\right\}$$

$$-\frac{1}{2\left(b^2-a^2\right)(1-\nu)}\left\{\left[(1-2\nu)-\frac{a^2}{r^2}\right]b^2\int_a^b \frac{\varepsilon_r^p-\varepsilon_\theta^p}{r}dr\right.$$

$$\left.+\left[1-(1-2\nu)\frac{a^2}{r^2}\right]\int_a^b r\left(\varepsilon_r^p+\varepsilon_\theta^p\right)dr\right\}, \tag{13}$$

$$\varepsilon_z = \frac{p_i a^2(1-2\nu)}{E\left(b^2-a^2\right)}+\frac{2}{b^2-a^2}\left[\int_a^b \alpha r T(r)dr-\int_a^b r\left(\varepsilon_r^p+\varepsilon_\theta^p\right)dr\right]. \tag{14}$$

3. <u>Non-dimensional Strain Equations for a Specific Temperature Distribution</u>

With prescribed surface temperatures T(b) = 0, T(a) = ΔT the axisymmetric temperature distribution for a sourcefree hollow cylinder is described by

$$T(r) = \Delta T \ln\frac{b}{r}\Big/\ln\frac{b}{a}\,. \tag{15}$$

Substitution of this specific temperature relationship into Eqs. (12), (13), and (14) and introducing the non-dimensional parameters

$$\rho = \frac{r}{a}, \qquad \rho_o = \frac{a}{b}, \tag{16}$$

gives:

$$\varepsilon_r = \frac{p_i}{E\left(\rho_o^2 - 1\right)}\left[(1-2\nu) - (1+\nu)\frac{\rho_o^2}{\rho^2}\right] - \frac{1+\nu}{1-\nu}\frac{\alpha \Delta T}{2}\left[\frac{1+2\ln\dfrac{\rho}{\rho_o}}{2\ln\rho_o} - \frac{\rho_o^2}{\rho^2\left(\rho_o^2-1\right)}\right]$$

$$+\frac{1-3\nu}{1-\nu}\frac{\alpha\Delta T}{2}\left[\frac{1}{2\ln\rho_o} - \frac{1}{\rho_o^2-1}\right] + \frac{1-2\nu}{1-\nu}\varepsilon_r^{*p}$$

$$+\frac{1-2\nu}{2(1-\nu)}\left[\int_1^{\rho}\frac{\varepsilon_r^{*p}-\varepsilon_\theta^{*p}}{\rho}d\rho - \frac{1}{\rho^2}\int_1^{\rho}\rho\left(\varepsilon_r^{*p}+\varepsilon_\theta^{*p}\right)d\rho\right]$$

$$+\frac{1}{2(1-\nu)\left(\rho_o^2-1\right)}\left[\left(1-\frac{1-2\nu}{\rho^2}\right)\int_1^{\rho}\rho\left(\varepsilon_r^{*p}+\varepsilon_\theta^{*p}\right)d\rho - \left((1-2\nu)-\frac{1}{\rho^2}\right)\rho_o^2\int_1^{\rho}\frac{\varepsilon_r^{*p}-\varepsilon_\theta^{*p}}{\rho}d\rho\right],$$

(17)

$$\varepsilon_\theta = \frac{p_i}{E\left(\rho_o^2 - 1\right)}\left[(1-2\nu) + (1+\nu)\frac{\rho_o^2}{\rho^2}\right] + \frac{1-3\nu}{1-\nu}\frac{\alpha\Delta T}{2}\left[\frac{1}{2\ln\rho_o} - \frac{1}{\rho_o^2-1}\right]$$

$$+\frac{1+\nu}{1-\nu}\frac{\alpha\Delta T}{2}\left[\frac{1-2\ln\dfrac{\rho}{\rho_o}}{2\ln\rho_o} - \frac{\rho_o^2}{\rho^2\left(\rho_o^2-1\right)}\right]$$

$$+\frac{1-2\nu}{2(1-\nu)}\left[\int_1^{\rho}\frac{\varepsilon_r^{*p}-\varepsilon_\theta^{*p}}{\rho}d\rho + \frac{1}{\rho^2}\int_1^{\rho}\rho\left(\varepsilon_r^{*p}+\varepsilon_\theta^{*p}\right)d\rho\right]$$

$$+\frac{1}{2\left(\rho_o^2-1\right)(1-\nu)}\left[\left(1+\frac{1-2\nu}{\rho^2}\right)\int_1^{\rho_o}\rho\left(\varepsilon_r^{*p}+\varepsilon_\theta^{*p}\right)d\rho - \left((1-2\nu)+\frac{1}{\rho^2}\right)\rho_o^2\int_1^{\rho_o}\frac{\varepsilon_r^{*p}-\varepsilon_\theta^{*p}}{\rho}d\rho\right]. \quad (18)$$

For *purely elastic* deformation the integral terms in the above equations for the principal strains are reduced to zero.

Thermal cycling is introduced into the problem by dividing each thermal cycle into two parts and considering these separately, assuming the following specification:

positive half-cycle: $\Delta T \neq 0$,

negative half-cycle: $\Delta T = 0$.

4. Relation Between Current Plastic Strains and Total Strains

The consideration *elastic-plastic* deformation requires the establishment of relationships between the plastic strains ε_{ij}^p occurring with a thermal half-cycle and the total strains ε_{ij}.

The current values of the total plastic strains are given by

$$\varepsilon_r^{*p} = \varepsilon_r^p + \sum \varepsilon_r^p, \qquad \varepsilon_\theta^{*p} = \varepsilon_\theta^p + \sum \varepsilon_\theta^p, \qquad \varepsilon_z^{*p} = \varepsilon_z^p + \sum \varepsilon_z^p, \qquad (20)$$

where $\sum\varepsilon_j$ designates the sum of the plastic strains incurred by the previous half-cycles.

As stress-strain law for the plastic range the Levy-von Mises plastic flow law for finite deformations is assumed to hold:

$$\frac{\varepsilon_r^p - \varepsilon_\theta^p}{\sigma_r - \sigma_\theta} = \frac{\varepsilon_r^p - \varepsilon_z^p}{\sigma_r - \sigma_z} = \frac{\varepsilon_\theta^p - \varepsilon_z^p}{\sigma_\theta - \sigma_z} = \lambda. \tag{21}$$

Introducing

$$\tilde{\varepsilon}_r = \varepsilon_r - \sum \varepsilon_r^p, \qquad \tilde{\varepsilon}_\theta = \varepsilon_\theta - \sum \varepsilon_\theta^p, \qquad \tilde{\varepsilon}_z = \varepsilon_z - \sum \varepsilon_z^p, \tag{22}$$

the stress-strain relations may be written in the following form:

$$\begin{aligned} \varepsilon_r &= \frac{1}{E}\left[\sigma_r - \nu(\sigma_\theta + \sigma_z)\right] + \varepsilon_r^p + \alpha T, \\ \varepsilon_\theta &= \frac{1}{E}\left[\sigma_\theta - \nu(\sigma_r + \sigma_z)\right] + \varepsilon_\theta^p + \alpha T, \\ \varepsilon_z &= \frac{1}{E}\left[\sigma_z - \nu(\sigma_r + \sigma_\theta)\right] + \varepsilon_z^p + \alpha T, \end{aligned} \tag{23}$$

and combining with Eq. (21) gives:

$$\frac{\varepsilon_r - \varepsilon_\theta}{\sigma_r - \sigma_\theta} = \frac{\varepsilon_r - \varepsilon_z}{\sigma_r - \sigma_z} = \frac{\varepsilon_\theta - \varepsilon_z}{\sigma_\theta - \sigma_z} = \Lambda, \tag{24}$$

where

$$\Lambda = \lambda + \frac{1+\nu}{E}. \tag{25}$$

Based on the von Mises yield condition

$$\sigma_V = \frac{1}{\sqrt{2}}\left[(\sigma_r - \sigma_\theta)^2 + (\sigma_r - \sigma_z)^2 + (\sigma_\theta - \sigma_z)^2\right]^{\frac{1}{2}} = 2k^* = \sigma_Y, \tag{26}$$

where σ_Y is the uniaxial yield stress, the following relations hold in terms of effective stresses and strains:

$$\lambda = \frac{3}{2}\frac{\varepsilon_v^p}{\sigma_V}, \qquad \Lambda = \frac{3}{2}\frac{\varepsilon_V}{\sigma_V}; \tag{27}$$

with

$$\varepsilon_v^p = \frac{\sqrt{2}}{3}\left[(\varepsilon_r^p - \varepsilon_\theta^p)^2 + (\varepsilon_r^p - \varepsilon_z^p)^2 + (\varepsilon_\theta^p - \varepsilon_z^p)^2\right]^{\frac{1}{2}}, \tag{28}$$

$$\varepsilon_V = \frac{\sqrt{2}}{3}\left[(\varepsilon_r - \varepsilon_\theta)^2 + (\varepsilon_r - \varepsilon_z)^2 + (\varepsilon_\theta - \varepsilon_z)^2\right]^{\frac{1}{2}}; \tag{29}$$

thus

$$\varepsilon_v^p = \varepsilon_V - \frac{2}{3}\frac{1+\nu}{E}\sigma_V. \tag{30}$$

In the elastic region: $\sigma_V < \sigma_Y$, $\varepsilon_v^p = 0$, and Eq. (30) reduces to:

$$\varepsilon_V = \frac{2}{3}\frac{1+\nu}{E}\sigma_V < \frac{2}{3}\frac{1+\nu}{E}\sigma_Y. \tag{31}$$

In the plastic region: $\sigma_V = \sigma_Y$, $\varepsilon_v^p \neq 0$, and from Eq. (30) follows:

$$\varepsilon_v^p = \varepsilon_V - \frac{2}{3}\frac{1+\nu}{E}\sigma_Y. \tag{32}$$

Therefore for any half-cycle:

$$\left.\begin{aligned} &\text{for } \varepsilon_V \le \frac{2}{3}\frac{1+\nu}{E}\sigma_Y ; \qquad \varepsilon_v^p = 0 \,, \\ &\text{for } \varepsilon_V \ge \frac{2}{3}\frac{1+\nu}{E}\sigma_Y ; \quad \varepsilon_v^p = \varepsilon_V - \frac{2}{3}\frac{1+\nu}{E}\sigma_Y \,. \end{aligned}\right\} \tag{33}$$

Elimination of the stress components from Eqs. (21) and (24) gives, in combination with Eqs. (27):

$$\frac{\varepsilon_r^p - \varepsilon_\theta^p}{\varepsilon_r - \varepsilon_\theta} = \frac{\varepsilon_r^p - \varepsilon_z^p}{\varepsilon_r - \varepsilon_z} = \frac{\varepsilon_\theta^p - \varepsilon_z^p}{\varepsilon_\theta - \varepsilon_z} = \frac{\varepsilon_v^p}{\varepsilon_V}, \tag{34}$$

wherefrom, under utilization of the incompressibility condition $\varepsilon_r^p + \varepsilon_\theta^p + \varepsilon_z^p = 0$, there follow the relations between the current plastic strains of a half-cycle and the total strains

$$\varepsilon_r^p = \frac{1}{3}\frac{\varepsilon_v^p}{\varepsilon_V}\left(2\varepsilon_r - \varepsilon_\theta - \varepsilon_z\right), \tag{35a}$$

$$\varepsilon_\theta^p = \frac{1}{3}\frac{\varepsilon_v^p}{\varepsilon_V}\left(2\varepsilon_\theta - \varepsilon_r - \varepsilon_z\right). \tag{35b}$$

With the relations given above the axisymmetric field of elastic-plastic plane strain deformation of a hollow cylinder under internal pressure and cyclic temperature variations at the internal boundary can be determined by an iterative procedure, - successively for each following half-cycle on the basis of the plastic strains accumulated during the previous half-cycles.

5. Iterative Procedure of Solution

An iterative procedure can now be used to determine the distributions of stress and strain during any half-cycle. The calculation steps of this procedure are as follows: At the start of each half-cycle $\varepsilon_r^p = \varepsilon_\theta^p = 0$ for all values of p; thus according to Eq. (20): $\varepsilon_r^{*p} = \sum \varepsilon_r^{*p}$; $\varepsilon_\theta^{*p} = \sum \varepsilon_\theta^{*p}$ are introduced as first approximation for the current values of the total plastic strains into Eqs. (17), (18), (19) which render the elastic-plastic total strains ε_r, ε_θ, ε_z.

Substitution of these values ε_r, ε_θ, ε_z into the Eqs. (22) and use of n_r, n_θ, n_z in Eq. (29) gives n_V, and Eq. (32) gives ε_V.

By means of Eqs. (35) new values for $\varepsilon_r^p, \varepsilon_\theta^p$ can be determined which are inserted in Eqs. (20), the resulting new values for $\varepsilon_r^{*p}, \varepsilon_\theta^{*p}$ are introduced into Eqs. (17), (18), (19) which now yield improved approximations for the elastic-plastic total strains.

The procedure is then repeated until there is no appreciable relative change in successive values of either the current plastic strains or the total strains for the half-cycle considered. When convergence has been reached to the desired degree of approximation, the stress components σ_r, σ_θ, σ_z may be calculated from Eqs. (4).

Starting with the first half-cycle for which $\sum \varepsilon_r^p = \sum \varepsilon_\theta^p = \sum \varepsilon_z^p = 0$, the iterative procedure must be carried out for each half-cycle in turn, utilizing the accumulated values for the plastic strains.

6. Bibliography

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