Course 22.314 “Structural Mechanics in Nuclear Power Technology”

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AXISYMMETRIC ELASTIC-PLASTIC PLANE STRAIN DEFORMATION OF A SOLID CIRCULAR CYLINDER OF TRESCA-MATERIAL UNDER UNIFORM HEAT GENERATION

Problem: Thermo-elastic-plastic deformation
Geometry: Solid cylinder, plane strain F(r)
Material: Isotropic; Tresca yield condition
Method: Analytical

1. Temperature Distribution

With uniform heat source density distribution

$$q_v(r) = Q = \text{const.}$$

Fourier’s heat conduction equation takes the following form in the steady state polarsymmetric (axisymmetric plane) case:

$$\frac{d^2\theta}{dr^2} + \frac{1}{r}\frac{d\theta}{dr} = -\frac{Q}{\lambda}, \qquad \text{or} \qquad \frac{1}{r}\frac{d}{dr}\left(r\frac{d\theta}{dr}\right) = -\frac{Q}{\lambda}, \tag{2}$$

where λ is the thermal conductivity.
Integration gives

$$r\frac{d\theta(r)}{dr} = -\frac{Q}{\lambda}\frac{r^2}{2} + C_1, \tag{3}$$

$$\theta(r) = -\frac{Qr^2}{4\lambda} + C_1 \ln r + C_2. \tag{4}$$

For a solid cylinder, $0 \le r \le r_o$, with Dirichlet’s boundary condition

$$\theta(r)\big|_{r=r_o} = \theta_o, \qquad \text{and} \qquad \left.\frac{d\theta(r)}{dr}\right|_{r=0} = 0 \tag{5a,b}$$

one obtains by utilizing the center condition Eq. (5b) in Eq. (3):

$$C_1 = 0, \tag{6}$$

and due to the boundary condition Eq. (5a) there follows from Eq. (4):

$$C_2 = \theta_o + \frac{Qr_o^2}{4\lambda} \tag{7}$$

Thus the following equation for the polarsymmetric temperature field is obtained:

$$T(r) = T_o + \frac{Q r_o^2}{4\lambda}\left(1 - \frac{r^2}{r_o^2}\right), \qquad 0 \le r \le r_o \tag{8}$$

Utilizing the temperature at the cylinder boundary as the reference temperature, i.e. $T_o = 0$, reduces Eq. (8) to:

$$T(r) = \frac{Q r_o^2}{4\lambda}\left(1 - \frac{r^2}{r_o^2}\right). \tag{8a}$$

Or, if otherwise the radial temperature variation is related to the temperature at the axis of the cylinder

$$T(r)\big|_{r=0} = T_o^c = \frac{Q r_o^2}{4\lambda}$$

the following expression for the temperature field is obtained:

$$T(r) = T_o^c\left(1 - \frac{r^2}{r_o^2}\right). \tag{8b}$$

2. <u>Elastic Behavior</u>

Under the assumption of constant (temperature independent) thermo-elastic materials data, the elastic strains in the axisymmetric case are described by the following relations:

$$\varepsilon_r^e = \frac{du_r}{dr} = \frac{1}{E}\left[\sigma_r - \nu(\sigma_\theta + \sigma_z)\right] + \alpha T(r), \tag{9a}$$

$$\varepsilon_\theta^e = \frac{u_r}{r} = \frac{1}{E}\left[\sigma_\theta - \nu(\sigma_z + \sigma_r)\right] + \alpha T(r), \tag{9b}$$

$$\varepsilon_z^e = \frac{dw}{dz} = \frac{1}{E}\left[\sigma_z - \nu(\sigma_r + \sigma_\theta)\right] + \alpha T(r). \tag{9c}$$

The equilibrium equation for the polarsymmetric case is:

$$\frac{d\sigma_r}{dr} + \frac{1}{r}\left(\sigma_r - \sigma_\theta\right) = 0. \tag{10}$$

In the central part of long cylindrical bodies there is a state of plane strain, i.e. ε_z = const., over the cross-section.

From these relations and simplifying assumptions the following general expressions for the thermoelastic stress components in the solid cylinder are obtained:

$$\sigma_r^e = -\frac{\alpha E}{(1-\nu)}\frac{1}{r^2}\int_{r=0}^{r} r T(r)\,dr + A + \frac{B}{r^2}, \tag{11a}$$

$$\sigma_\theta^e = \frac{\alpha E}{(1-\nu)}\frac{1}{r^2}\int_{r=0}^{r} r T(r)\,dr - \frac{\alpha E T(r)}{1-\nu} + A + \frac{B}{r^2}, \tag{11b}$$

$$\sigma_z^e = -\frac{\alpha E T(r)}{1-\nu} + C. \tag{11c}$$

Substituting the particular temperature distribution as given by Eq. (8b) into these equations and determination of the integration constants from the boundary conditions, - i.e. at the cylinder surface $\sigma_r = 0$, at the cylinder axis extreme values of the stress components occur, the end faces of the cylinder are free, -gives the following specific relations for the stress components:

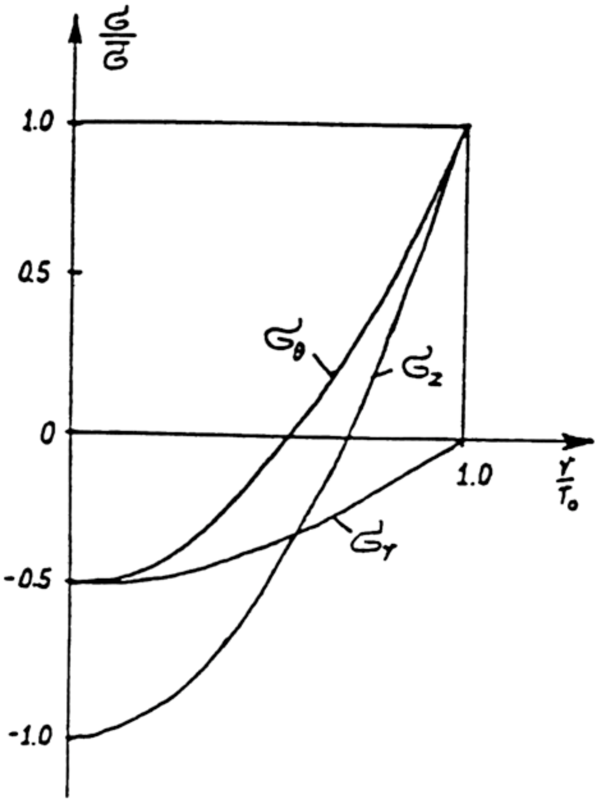
$$\sigma_r^e = \frac{E\alpha\theta_o^c}{4(1-\nu)}\left(\frac{r^2}{r_o^2}-1\right)+A^*, \tag{12a}$$

$$\sigma_\theta^e = \frac{E\alpha\theta_o^c}{4(1-\nu)}\left(3\frac{r^2}{r_o^2}-1\right)+A^*, \tag{12b}$$

$$\sigma_z^e = \frac{E\alpha\theta_o^c}{4(1-\nu)}\left(4\frac{r^2}{r_o^2}-1\right)+B^*, \tag{12c}$$

Where A* and B* are new constants introduced in order to cope with the elastic-plastic interface conditions. Figure 1 shows the variation of these stress components as a function of radius.

Figure 1



3. Initial Plastic Yield

It is assumed here that the material obeys Tresca's yield criteria of constant maximum shear stress (i.e. maximum principal stress difference) which in the cylindrical coordinate system is given by the relations

$$
\begin{aligned}
\left|\sigma_\mathrm{r} - \sigma_\theta\right| &= 2k', \qquad \text{for} \quad \begin{cases} \sigma_\mathrm{r} > \sigma_\mathrm{z} > \sigma_\theta\,, \\ \sigma_\theta > \sigma_\mathrm{z} > \sigma_r\,, \end{cases} \\
\left|\sigma_\theta - \sigma_z\right| &= 2k', \qquad \text{for} \quad \begin{cases} \sigma_\theta > \sigma_\mathrm{r} > \sigma_z\,, \\ \sigma_z > \sigma_\mathrm{r} > \sigma_\theta\,, \end{cases} \\
\left|\sigma_\mathrm{z} - \sigma_r\right| &= 2k', \qquad \text{for} \quad \begin{cases} \sigma_\mathrm{z} > \sigma_\theta > \sigma_r\,, \\ \sigma_r > \sigma_\theta > \sigma_z\,, \end{cases}
\end{aligned}
\tag{13}
$$

where

$$
k' = \frac{\sigma_o}{2} \tag{14}
$$

is the yield limit for pure shear loading and σ_o in the yield in uniaxial tension or compression. (Note: The Tresca yield hypothesis neglects the influence of the intermediate principal stress component.)

Examination of Eqs. (12), as illustrated by Fig. 1 shows that the maximum difference of the principal stress components occurs at the surface of the solid cylinder $r = r_o$, so that with increasing temperature gradient (due to an increase in the heat source density) plastic yielding is starting from there.

Combining Eqs. (12) with Tresca's yield condition, Eq. (13), gives an expression for the temperature difference at which <u>initial yielding</u> occurs at the surface of the solid cylinder:

$$
2k' = \left|\sigma_\theta - \sigma_r\right|_{r=r_o} = \left|\sigma_z - \sigma_r\right|_{r=r_o} = \frac{E\alpha\vartheta_o^c}{2(1-\nu)}\,. \tag{15}
$$

4. <u>Elastic-Plastic Behavior</u>

With further increasing temperature difference between center and surface of the cylinder a plastic ring zone, growing inwards, is produced. The inner radius of the plastic zone is designated by r*.

Under the assumption that <u>no work-hardening occurs and under the provisional assumption that in the plastic zone the value of the axial stress σ_Z remains between the values of σ_θ and σ_r</u> the following relation is valid for the plastic domain:

$$
\left|\sigma_\theta - \sigma_r\right| = 2k', \qquad r^* \le r \le r_o\,. \tag{16}
$$

Substituting this relation into the equilibrium equation, Eq. (10), gives a differential equation for the radial plastic stress component:

$$
\frac{d\sigma_r}{dr} = \frac{2k'}{r} \tag{17}
$$

which has the general solution:

$$
\sigma_r = 2k' \int_{r_o}^{r} \frac{dr}{r} = -2k' \ln\frac{r_o}{r} + C_1\,. \tag{18}
$$

From the boundary condition $\sigma_r(r)\big|_{r=r_a} = 0$ there follows $C_1 = 0$, thus:

$$
\sigma_r^p = -2k' \ln\frac{r_o}{r}\,, \qquad\qquad r^* \le r \le r_o\,. \tag{19}
$$

The circumferential plastic stress σ_θ is obtained by substituting Eqs. (17) and (19) into the equilibrium equation, Eq. (10):

$$\sigma_\theta^p = 2k^*\left[1-\ln\frac{r_o}{r}\right], \qquad r^* \le r \le r_o. \tag{20}$$

For the interface between elastic and plastic zone according to Eqs. #### and (12a,b) the following relation holds:

$$2k^* = \left|\sigma_\theta - \sigma_r\right|_{r=r^*} = \frac{E\alpha\Delta T_o^c}{2(1-\nu)}\frac{r^{*2}}{r_0^2} \qquad \text{or} \qquad \frac{r_0^2}{r^{*2}} = \frac{E\alpha\Delta T_o^c}{4(1-\nu)k^*}. \tag{21}$$

For any given value of the center temperature ΔT_o^c, Eq. (21) allows the associated value of r* to be calculated. The constants in the Eqs. (12) can be determined by utilizing the equilibrium conditions of radial and circumferential stresses, respectively, at the interface, that means by equating of Eqs. (12a) and (19) and of Eqs. (12b) and (20). In this way the following expressions are obtained for the elastic stress components σ_r^e and σ_θ^e:

$$\begin{aligned} \sigma_r^e &= k^*\left[-1+\frac{r^2}{r^{*2}}-2\ln\frac{r_o}{r^*}\right], \\ & \qquad\qquad\qquad\qquad\qquad 0 \le r \le r^*, \\ \sigma_\theta^e &= k^*\left[-1+3\frac{r^2}{r^{*2}}-2\ln\frac{r_o}{r^*}\right], \end{aligned} \tag{22}$$

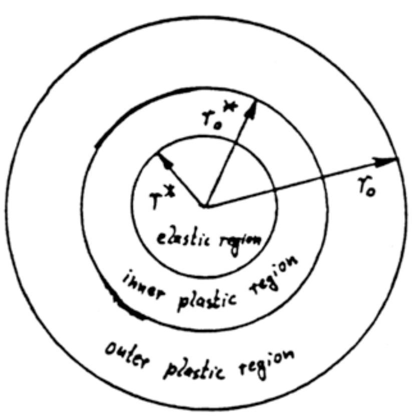
While the determination of the radial and tangential components of the #### field of the partially plasticized cylinder could be done solely by means of equilibrium conditions, the determination of the axial stress component requires stress-strain relationships which in the plastic range are given by the plastic potential yield law:

$$\dot{\varepsilon}_r^p = \lambda\frac{\partial f}{\partial \sigma_r}, \qquad \dot{\varepsilon}_\theta^p = \lambda\frac{\partial f}{\partial \sigma_\theta}, \qquad \dot{\varepsilon}_z^p = \lambda\frac{\partial f}{\partial \sigma_z}, \tag{23}$$

(the parameter λ is often designated by $\dot{\lambda}$ or Λ), as well as a yield condition, which here has been chosen to be Tresca's yield criterion as #### by Eqs. (13).

ELLINGTON has shown that the yield zone of the uniformly heat generating, partially plasticized solid cylinder consists of two regions:

a) an outer plastic zone $r_o^* \le r \le r_o$ where $\sigma_\theta = \sigma_z > \sigma_r$, and
b) an inner plastic zone $r^* \le r \le r_o^*$ where $\sigma_\theta > \sigma_z > \sigma_r$.



For the outer plastic zone there follows from Eq. (20)

$$\sigma_z^p = \sigma_\theta^p = 2k^*\left(1 - \ln\frac{r_o}{r}\right), \qquad r_o^* \le r \le r_o. \qquad (24)$$

For the inner plastic zone from the plastic yield law, Eq. (23) with $\tau = \left(\sigma_\theta - \sigma_r\right) = ###$ there follow the relations:

$$\varepsilon_\theta^p = -\varepsilon_r^p, \qquad \varepsilon_z^p = 0, \qquad (25)$$

thus the axial strain in this domain is pure elastic:

$$\varepsilon_z = \varepsilon_z^e + \alpha T.$$

Utilizing the stress-strain relation Eq.#### and the equilibrium equation, Eq. (10), yield the relation:

$$\sigma_z = E\varepsilon_z - E\alpha T + \nu\left(2\sigma_r + r\frac{d\sigma_r}{dr}\right), \qquad 0 \le r \le r_o^* \qquad (26)$$

In the case of free, unloaded end faces of the cylinder:

$$\int_0^{r_o} \sigma_z r\,dr = \int_0^{r_o^*}\left[E\varepsilon_z - E\alpha T + \frac{\nu}{r}\frac{d}{dr}\left(r^2\sigma_r\right)\right] r\,dr + \int_{r_o^*}^{r_o} 2k^*\left(1 - \ln\frac{r_o}{r}\right) r\,dr = ### \qquad (27)$$

where from by integration and utilization of Eqs. (19) and (21):

$$\frac{E\varepsilon_z}{2k^*} = (1-\nu)\frac{r_o^2}{r^{*2}}\left(2 - \frac{r_o^{*2}}{r_o^2}\right) - (1-2\nu)\ln\frac{r_o}{r_o^*} + \frac{1}{2}\left(1 - \frac{r_o^2}{r_o^{*2}}\right). \qquad (28)$$

To obtain σ_z, this expression has to be inserted, together with Eqs.#### into Eq. (26). The determination of the interface radius r_o^*, which separates the two different plastic zones, can be done by means of the condition of the continuity of the axial stresses σ_z at the interface. Equations (24) and (26) at $r = r_o^*$ and making use of Eqs. (21) and (28) gives the relation:

$$\frac{r_o^2}{r_o^{*2}} = \frac{1}{2}\left[-(1-2\nu) + \left\{(1-2\nu)^2 + 8(1-\nu)\frac{r_o^2}{r^{*2}}\right\}^{1/2}\right]. \qquad (29)$$

For the calculation of the radial displacement u_r, -under the assumption of plastic incompressibility-, the cubic dilatation relation can be used which for an elastic-plastic material and cylindrical geometry is given by:

$$\frac{du_r}{r} + \frac{u_r}{r} + \varepsilon_z = \frac{1-2\nu}{E}\left(\sigma_r + \sigma_\theta + \sigma_z\right) + 3\alpha T. \qquad (30)$$

For the elastic core and the inner plastic zone the stress components σ_z and σ_r may be eliminated from Eq. (30) by means of Eq. (26) and the equilibrium equation, Eq. (10). Thereby, the following differential equation for the radial displacement u_r is obtained:

$$\frac{1}{r}\frac{d}{dr}\left(ru_r\right) = (1+\nu)\frac{1-2\nu}{E}\frac{1}{r}\frac{d}{dr}\left(r^2\sigma_r\right) - 2\nu\varepsilon_z + 2(1+\nu)\alpha T(r) \qquad (31)$$

which by use of the temperature field equation Eq. (8b) and integration gives:

$$ru_r = (1+\nu)\frac{1-2\nu}{E}r^2\sigma_r - \nu\varepsilon_z r^2 + 2(1+\nu)\alpha T_o^c\frac{r^2}{2}\left(1 - \frac{r^2}{2r_o^2}\right). \qquad (32)$$

The tangential strain for the central elastic region is obtained by introducing the radial stress relation according to Eq. (22) into Eq. (32):

$$\varepsilon_\theta^e = \frac{u_r}{r} = (1+\nu)\frac{2k'}{E}\left[2(1-\nu)\frac{r_o^2}{r'^2} - (1-2\nu)\left(\frac{1}{2} + \ln\frac{r_o}{r'}\right) - \frac{r^2}{2r'^2}\right] - \nu\varepsilon_z \tag{33}$$

the tangential strain for the <u>inner plastic region</u> is obtained by substitution of the radial stress relation according to Eq. (19):

$${}^i\varepsilon_\theta = (1+\nu)\frac{2k'}{E}\left[2(1-\nu)\frac{r_o^2}{r'^2} - (1-2\nu)\ln\frac{r_o}{r'} - (1-\nu)\frac{r^2}{r'^2}\right] - \nu\varepsilon_z, \qquad r' \le r \le r_o' \tag{34}$$

The tangential strain in the <u>outer plastic region</u> may be obtained from elastic-plastic cubic dilatation condition for plastic incompressibility, Eq. (30), which by utilizing the equilibrium condition Eq. (10) can be written in the form:

$$\frac{1}{r}\frac{d}{dr}(ru_r) = \frac{1-2\nu}{E}\frac{1}{r}\frac{d}{dr}\left(r^2\sigma_r\right) - \varepsilon_z + 3\alpha T + \frac{1-2\nu}{E}\sigma_z$$

or

$$ru_r = \frac{1-2\nu}{E}\left(r^2\sigma_r\right) - \int_0^r(\varepsilon_z - 3\alpha T)rdr + \frac{1-2\nu}{E}\int_0^r \sigma_z rdr, \tag{35}$$

from which, by using Eq. (24),

$$\varepsilon_\theta = (1-2\nu)\frac{k'}{2E}\left[1 - \frac{r_o^2}{r^2} - 6\ln\frac{r_o}{r}\right] + (1-\nu)\frac{6k'}{E}\left[\frac{r_o^2}{r'^2} - \frac{r^2}{2r'^2}\right] - \frac{\varepsilon_z}{2}, \qquad r_o' \le r \le r_o. \tag{36}$$

Under the specified assumption that the value of the axial stress σ_z is intermediate to the tangential and the radial stress, the above solution is valid up to the start of yield at the axis of the cylinder, r = 0, where according to Eqs. (22): $\sigma_r = \sigma_\theta$. The condition for the start of yielding at the axis of the cylinder thus is:

$$\left|\sigma_r - \sigma_z\right|_{r=0} = \left|\sigma_\theta - \sigma_z\right|_{r=0} = 2k'. \tag{37}$$

As ELLINGTON shows, beyond this state of plastification the given solution is no longer generally valid.

5. <u>Bibliography</u>

Principal Reference:

J.P. Ellington: Thermal Strains in a Heat Producing Elastic-Plastic Cylinder. DEG Report 307 (R), U.K.A.E.A., Risley, Warrington, Lancashire, (1961).

Auxiliary Reference:

A. Mendelson: <u>Plasticity: Theory and Application.</u> The Mcmillan Company, New York, (1968).

Further recommended reading:

T.B. Kammash, S.A. Murch, P.M. Naghdi: The Elastic-Plastic Cylinder Subjected to Radially Distributed Heat Source, Lateral Pressure and Axial Force with Application to Nuclear Reactor Fuel Elements. Journal of the Mechanics and Physics of Solids, 8 (1960) No.1, p. 1-25.

B.W. Shaffer: The Elastic-Plastic Stresses Within a Tube Generating Internal Heat Uniformly. Report APEX-659, General Electric, Nuclear Materials and Propulsion Operation, Cincinnati, Ohio, (1961).