Course 22.314 “Structural Mechanics in Nuclear Power Technology”

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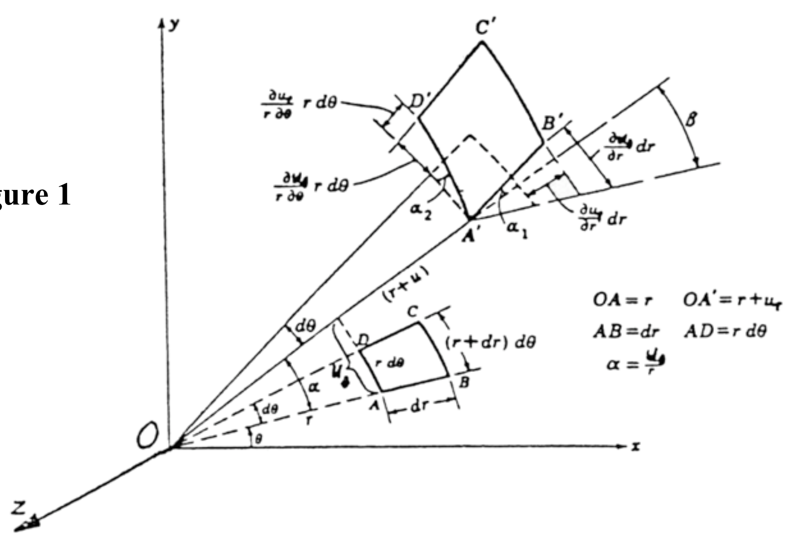
APPENDIX: CONTINUUM EQUATIONS FOR THREE-DIMENSIONAL ELASTIC AND THERMOELATIC STRESS-STRAIN FIELDS IN CYLINDRICAL COORDINATES

Contents

1. Strain-Displacement Relations in Cylindrical Coordinates

The consideration of the relations between strain and displacement in cylindrical coordinates r, θ, z for reasons of simplicity is first confined to the consideration of the two-dimensional case where it is assumed that all points which are in the (r, θ)-plane before deformation remain in the same plane during deformation. This means that the problem is initially attacked in polar coordinates only, assuming that the displacements components in the r- and θ-direction, u_r and u_θ, are independent of the axial (z-direction) displacement component w, with the influence this latter component to be considered later.

Figure 1



The derivation is based on the consideration of an infinitesimal ##### element ABCD (Fig. 1) which by deformation of the body is deformed displaced to A'B'C'D'. By definition, the principal strain in the radial direction is given by:

$$\varepsilon_r = \frac{\overline{A'B'} - \overline{AB}}{\overline{AB}} \qquad (1)$$

thus

$$\overline{A'B'} = (1+\varepsilon_r)\overline{AB} = (1+\varepsilon_r)dr\,. \qquad (2)$$

Referring to Fig. 1, the small variations of the radial and tangential displacements from A to B may be obtained by multiplying the rates of change of u_r and u_θ with respect to r at A by the length of the infinitesimal element $\overline{AB} = dr$. Herewith the following trigonometric relationship obtained:

$$\left(\overline{A'B'}\right)^2 = (1+\varepsilon_r)^2 (dr)^2 = \left(dr + \frac{\partial u_r}{\partial r}dr\right)^2 + \left(\frac{\partial u_\theta}{\partial r}dr\right)^2, \qquad (3)$$

where the partial derivatives are used for the reason that u_r and u_θ vary as functions of both r and θ.

In the so-called small deformation theory the radial strain ε_r and the derivatives of the displacements, $\partial u_r / \partial r$ and $\partial u_\theta / \partial r$ are considered to be small. Therefore, the squares and products of these quantities may be neglected in comparison with the quantities themselves, so that under these conditions Eq. (3) reduces to:

$$\varepsilon_r = \frac{\partial u_r}{\partial r}\,. \qquad (4)$$

Correspondingly, the principal strain in the tangential direction

$$\varepsilon_\theta = \frac{\overline{A'D'} - \overline{AD}}{\overline{AD}} \qquad (5)$$

gives the relation

$$\overline{A'D'} = (1+\varepsilon_\theta)\overline{AD} = (1+\varepsilon_\theta)rd\theta, \qquad (6)$$

hence

$$\left(\overline{A'D'}\right)^2 = (1+\varepsilon_\theta)^2 (rd\theta)^2 = \left[(r+u_r)d\theta + \frac{\partial u_\theta}{\partial \theta}rd\theta\right]^2 + \left[\frac{\partial u_r}{r\partial \theta}\right]^2$$

or

$$2\varepsilon_\theta + \varepsilon_\theta^2 = 2\left(\frac{u_r}{r} + \frac{\partial u_\theta}{r\partial \theta} + \frac{u_r}{r}\frac{\partial u_\theta}{r\partial \theta}\right) + \left(\frac{u_r}{r}\right)^2 + \left(\frac{\partial u_\theta}{r\partial \theta}\right)^2 + \left(\frac{\partial u_r}{r\partial \theta}\right)^2. \qquad (7)$$

Under the conditions of small deformation theory, Eq. (7) reduces to

$$\varepsilon_\theta = \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta}\,. \qquad (8)$$

The shear strains are defined as the angles of distortion of the original angles of the element. Thus the shear strain $\gamma_{r\theta}$ is by definition (see Fig. 1)

$$\gamma_{r\theta} = \beta_1 + \beta_2\,, \qquad (9)$$

where β_1 and β_2 denote the angles of inclination between $\overline{AB}$ and $\overline{A'B'}$, resp. between $\overline{AD}$ and $\overline{A'D'}$. Under the assumption that the angular changes β_1 and β_2 may be considered as being small and assuming that $\partial u_r / \partial r$ and $\partial u_\theta / \partial r$ are much smaller than unity,

$$\beta_1 + \omega \approx \tan(\beta_1 + \omega) = \frac{(\partial u_\theta / \partial r)dr}{(1 + \partial u_r / \partial r)dr} \approx \frac{\partial u_\theta}{\partial r}$$

where ω is the rotational angle of the element due to deformation, we obtain:

$$\beta_1 = \frac{\partial u_\theta}{\partial r} - \omega = \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}. \tag{10}$$

Similarly the angular change β_2 can be determined as

$$\beta_2 \approx \tan\beta_2 = \frac{(\partial u_r / \partial r) r d\theta}{\left(1 + \frac{u_r}{r} + \frac{\partial u_\theta}{r\partial\theta}\right) r d\theta} \approx \frac{\partial u_r}{r\partial\theta}. \tag{11}$$

Substitution of Eqs. (10) and (11) into Eq. (9) gives

$$\gamma_{r\theta} = \frac{1}{r}\frac{\partial u_r}{\partial\theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}. \tag{12}$$

Equations (4), (8) and (12) give the two-dimensional strain-displacement relations in polar coordinates.

In the general three-dimensional case in cylindrical geometry, when z is the axial direction, there are a further normal strain ε_z and two shear strains $\gamma_{\theta z}$ and γ_{zr}. With w denoting the displacement component in the axial direction, the complete set of strain-displacement relations in cylindrical coordinates is:

$$\begin{aligned}
&\varepsilon_r = \frac{\partial u_r}{\partial r}, \qquad \varepsilon_\theta = \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial\theta}, \qquad \varepsilon_z = \frac{\partial w}{\partial z}, \\
&\gamma_{r\theta} = 2\varepsilon_{r\theta} = \frac{1}{r}\frac{\partial u_r}{\partial\theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}, \\
&\gamma_{\theta z} = 2\varepsilon_{\theta z} = \frac{\partial u_\theta}{\partial z} + \frac{1}{r}\frac{\partial w}{\partial\theta}, \qquad \gamma_{zr} = 2\varepsilon_{zr} = \frac{\partial u_r}{\partial z} + \frac{\partial w}{\partial r}.
\end{aligned} \tag{13}$$

The relations between displacements and rotations of the element of the deformed body are:

$$w_r = \frac{1}{2}\left(\frac{1}{r}\frac{\partial w}{\partial\theta} - \frac{\partial u_\theta}{\partial z}\right), \quad w_\theta = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial z} - \frac{\partial w}{\partial r}\right), \quad w_z = \frac{1}{2r}\left(\frac{\partial(ru_\theta)}{\partial r} - \frac{\partial u_r}{\partial\theta}\right). \tag{14}$$

2. Strain-Compatibility Equations in Cylindrical Coordinates

Equations (13) express six strain components in terms of the three displacement components u_r, u_θ and u_z. This implies that, if the strain components were arbitrarily prescribed, the six equations cannot, in general, be expected to yield single-valued continuous solutions for u_r, u_θ and u_z. Therefore, conditions of noncontradiction must be formulated to ensure the compatibility of the strain components in order to give single valued continuous solutions for the displacements.

These conditions, which are called strain-compatibility equations, are obtained from the strain displacement equations, Eqs. (13), by elimination of the displacement components. The resulting strain-compatibility equations in cylindrical coordinates are:

$$\frac{\partial^2\varepsilon_r}{\partial\theta\partial z} = \frac{1}{r}\frac{\partial}{\partial r}\left(r^2\frac{\partial\varepsilon_{\theta z}}{\partial r} + r\frac{\partial\varepsilon_{zr}}{\partial\theta} + r^2\frac{\partial\varepsilon_{r\theta}}{\partial z}\right) - \frac{2}{r}\frac{\partial\varepsilon_{zr}}{\partial\theta},$$

$$\frac{\partial^2(r\varepsilon_\theta)}{\partial z\partial r} - \frac{\partial\varepsilon_r}{\partial z} = \frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{\partial(r\varepsilon_{\theta z})}{\partial r} - \frac{\partial\varepsilon_{zr}}{\partial\theta} + r\frac{\partial\varepsilon_{r\theta}}{\partial z}\right),$$

$$\frac{\partial^2\varepsilon_z}{\partial r\partial\theta} - \frac{1}{r}\frac{\partial\varepsilon_z}{\partial\theta} = \frac{\partial}{\partial z}\left(r\frac{\partial\varepsilon_{\theta z}}{\partial r} + r\frac{\partial\varepsilon_{zr}}{\partial\theta} - r\frac{\partial\varepsilon_{r\theta}}{\partial z}\right) - \frac{\partial\varepsilon_{\theta z}}{\partial z},$$

$$2\frac{\partial^2(r\varepsilon_{r\theta})}{\partial r\partial\theta} = \frac{\partial}{\partial r}\left(r^2\frac{\partial\varepsilon_\theta}{\partial r}\right) - r\frac{\partial\varepsilon_r}{\partial r} + \frac{\partial^2\varepsilon_r}{\partial\theta^2},$$

$$2\left(\frac{\partial^2\varepsilon_{\theta z}}{\partial\theta\partial z} + \frac{\partial\varepsilon_{zr}}{\partial z}\right) = \frac{1}{r}\frac{\partial^2\varepsilon_z}{\partial\theta^2} + r\frac{\partial^2\varepsilon_\theta}{\partial z^2} + \frac{\partial\varepsilon_z}{\partial r},$$

$$2\frac{\partial^2\varepsilon_{zr}}{\partial z\partial r} = \frac{\partial^2\varepsilon_r}{\partial z^2} + \frac{\partial^2\varepsilon_z}{\partial r^2} \qquad (15)$$

where

$$\varepsilon_{r\theta} = \frac{1}{2}\gamma_{r\theta}, \qquad \varepsilon_{\theta z} = \frac{1}{2}\gamma_{\theta z}, \qquad \varepsilon_{zr} = \frac{1}{2}\gamma_{zr}. \qquad (15a)$$

3. Stress Equilibrium Equations in Cylindrical Coordinates

The stress-equilibrium equations in cylindrical coordinates r, θ and z are derived from the consideration of the conditions of equilibrium of an infinitesimal volume element

$$dV = r\,dr\,d\theta\,dz$$

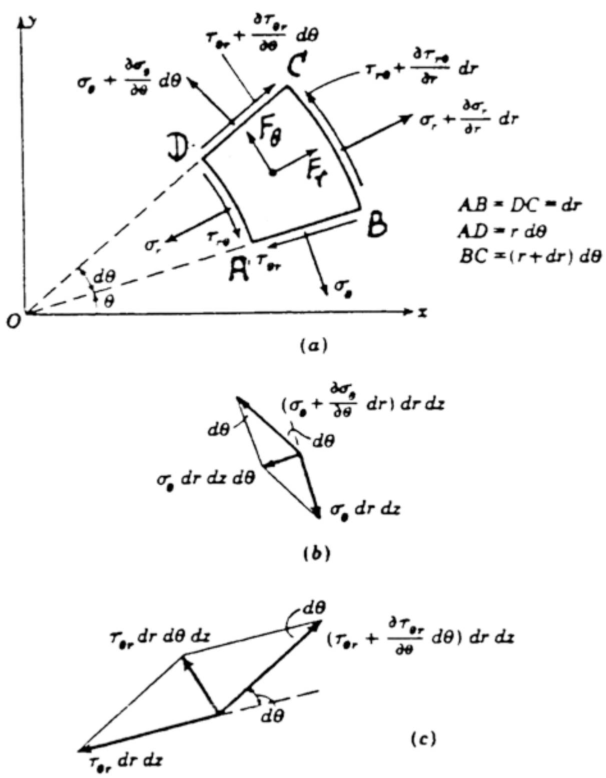


Figure 2

Since the object is to analyze the variation of stresses, infinitesimal changes in the stress components are taken into account as indicated in Fig. 2. It is assumed that each stress component acts at the centroid of its side. Hence, the corresponding force is the product

of the stress and the area of the side. Body forces per unit of volume are assumed to act at the centroid of the infinitesimal volume element, their components F_r, F_θ, and F_z are acting in the coordinate directions.
For simplicity, the equilibrium considerations are first confined to the (r, θ)-plane with the contributions from forces acting in the (z, r)- and (z, θ)-planes to be incorporated thereafter.
Considering the two-dimensional state of stress in the (r, θ)-plane, for a slice thickness dz, the condition of equilibrium of forces acting on an infinitesimal element, as shown in plane (r, θ) view in Fig. 2a, in the radial direction r is:

$$F_r r dr d\theta dz + \left(\sigma_r + \frac{\partial \sigma_r}{\partial r} dr\right)(r + dr) d\theta dz - \sigma_r r d\theta dz + \left(\tau_{\theta r} + \frac{\partial \tau_{\theta r}}{\partial \theta}\right) drdz - \tau_{\theta r} drdz - \sigma_\theta drdzd\theta = 0$$

(17)
where the last term on the left side represents the radial component of the circumferential forces $\sigma_\theta drdz$ and $\left(\sigma_\theta + \frac{\partial \sigma_\theta}{\partial \theta} d\theta\right) drdz$ as shown in Fig. 2b.

The condition of the equilibrium of forces acting on the infinitesimal element in the tangential direction θ is:

$$F_\theta r dr d\theta dz + \left(\sigma_\theta + \frac{\partial \sigma_\theta}{\partial \theta} d\theta\right) drdz - \sigma_\theta drdz + \left(\tau_{r\theta} + \frac{\partial \tau_{r\theta}}{\partial r} dr\right)(r + dr) d\theta dz - \tau_{r\theta} r d\theta dz + \tau_{\theta r} dr d\theta dz = 0$$

, (18)
where the last term on the left side is the tangential component of the forces $\tau_{\theta r} drdz$ and $\left(\tau_{\theta r} + \frac{\partial \tau_{\theta r}}{\partial \theta} d\theta\right) drdz$ as shown in Fig. 2c.

Consideration of the three-dimensional state of equilibrium of forces acting on the infinitesimal volume element shows that the forces acting in the (axial) z-direction do not contribute any component to the stress-state in the r- and θ-directions, and vice versa the forces in the r- and θ-directions do not contribute any component to the stress-state in the z-direction.

Therefore, for completing the above planar equilibrium equations for the three-dimensional case, it is only necessary to add to the left side of Eq. (17) the term

$$\left[\left(\tau_{zr} + \frac{\partial \tau_{zr}}{\partial z} dz\right) - \tau_{zr}\right]\left(r + \frac{dr}{2}\right) dr d\theta \qquad (19)$$

and to add to the left side of Eq. (18) the term

$$\left[\left(\tau_{z\theta} + \frac{\partial \tau_{zr}}{\partial z} dz\right) - \tau_{z\theta}\right]\left(r + \frac{dr}{2}\right) dr d\theta. \qquad (20)$$

The condition of equilibrium in the axial direction z is:

$$F_z r dr d\theta dz + \left(\sigma_z + \frac{\partial \sigma_z}{\partial z} dz\right) r d\theta dz - \sigma_z rdrdz + \left(\tau_{\theta z} + \frac{\partial \tau_{\theta z}}{\partial \theta} d\theta\right) drdz - \tau_{\theta z} drdz + \left(\tau_{rz} + \frac{\partial \tau_{rz}}{\partial r} dr\right)(r + dr) d\theta dz + \tau_{rz} rd$$

.(21)

The planar stress-equilibrium equations, Eqs. (17) and (18), completed by the respective terms, Eqs. (19) and (20), and Eq. (21), on eliminating differential terms of higher order, reduce to:

$$\frac{\partial \sigma_r}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\partial \tau_{rz}}{\partial z} + \frac{\sigma_r - \sigma_\theta}{r} + F_r = 0 \tag{22a}$$

$$\frac{\partial \tau_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_\theta}{\partial \theta} + \frac{\partial \tau_{\theta z}}{\partial z} + \frac{2}{r}\tau_{r\theta} + F_\theta = 0 \tag{22b}$$

$$\frac{\partial \tau_{rz}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta z}}{\partial \theta} + \frac{\partial \sigma_z}{\partial z} + \frac{1}{r}\tau_{rz} + F_z = 0. \tag{22c}$$

In arriving at the reduced equilibrium equations, the conditions

$$\tau_{r\theta} = \tau_{\theta r}, \qquad \tau_{\theta z} = \tau_{z\theta}, \qquad \tau_{zr} = \tau_{rz}, \tag{23}$$

which follow from a consideration of the moment-equilibrium of all the forces acting on the volume element, have been utilized.

4. Linear Elastic Stress-Strain Relations in Cylindrical Coordinates

The stress-strain relations in the cylindrical coordinate system for isotropic linear elastic material with constant thermoelastic materials properties in explicit form for the strain components and in terms of the elasticity parameters, Young's modulus E, Poisson's ratio ν, and shear modulus G, are:

$$\begin{aligned}
\varepsilon_r &= \frac{\partial u_r}{\partial r} = \frac{1}{E}\left[\sigma_r - \nu(\sigma_\theta + \sigma_z)\right] + \alpha\Delta T,\\
\varepsilon_\theta &= \frac{u_r}{r} = \frac{1}{E}\left[\sigma_\theta - \nu(\sigma_r + \sigma_z)\right] + \alpha\Delta T,\\
\varepsilon_z &= \frac{\partial w}{\partial z} = \frac{1}{E}\left[\sigma_z - \nu(\sigma_r + \sigma_\theta)\right] + \alpha\Delta T,\\
\varepsilon_{r\theta} &= \frac{1}{2G}\tau_{r\theta}, \varepsilon_{\theta z} = \frac{1}{2G}\tau_{\theta z}, \varepsilon_{rz} = \frac{1}{2G}\tau_{rz}.
\end{aligned} \tag{24}$$

The same relations written in explicit form for the stress components and in terms of Lame's constants λ and μ are:

$$\begin{aligned}
\sigma_r &= \lambda\varepsilon_{kk} + 2\mu\varepsilon_r - (3\lambda + 2\mu)\alpha\Delta T,\\
\sigma_\theta &= \lambda\varepsilon_{kk} + 2\mu\varepsilon_\theta - (3\lambda + 2\mu)\alpha\Delta T,\\
\sigma_z &= \lambda\varepsilon_{kk} + 2\mu\varepsilon_z - (3\lambda + 2\mu)\alpha\Delta T,\\
\tau_{r\theta} &= 2\mu\varepsilon_{r\theta}, \tau_{\theta z} = 2\mu\varepsilon_{\theta z}, \tau_{rz} = 2\mu\varepsilon_{rz},
\end{aligned} \tag{25}$$

where ε_{kk} denotes the volume dilatation which is derived in the following #####
The Lame's constants are defined as follows:

$$\mu = \frac{E}{2(1+\nu)} = G, \qquad \lambda = \frac{\nu E}{(1+\nu)(1-2\nu)} = \frac{2\nu G}{1-2\nu}. \tag{26}$$

The bulk modulus K (also designated as compression modulus)

$$K = \frac{2}{3}\frac{1+\nu}{1-2\nu}G = \frac{E}{3(1-2\nu)} \tag{27}$$

is related to the Lame's constants by: $3\lambda + 2\mu = 3K$, and Young's modulus and Poisson's ratio may be expressed as:

$$E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu} = \frac{9GK}{3K+G}, \tag{28}$$

$$\nu = \frac{\lambda}{2(\lambda+\mu)} = \frac{3K - 2G}{6K + 2G}. \tag{29}$$

The relation between the volume dilatation ε_{kk}

$$\varepsilon_{kk} = \varepsilon_r + \varepsilon_\theta + \varepsilon_z = \frac{1}{r}\frac{\partial(ru_r)}{\partial r} + \frac{1}{r}\frac{\partial u_q}{\partial \theta} + \frac{\partial w}{\partial z} \tag{30}$$

and the sum of the normal σ_{kk}

$$\sigma_{kk} = \sigma_r + \sigma_\theta + \sigma_z \tag{31}$$

follows from Eqs. (24)

$$\varepsilon_{kk} = \frac{\sigma_{kk}}{3K} + 3\alpha\Delta T. \tag{32}$$

Utilizing the elastic parameters G and ν and the volume dilatation ε_{kk}, the relations between stresses and strains, Eqs. (25), may also be expressed as follows:

$$\left.\begin{aligned}
\sigma_r &= 2G\left[\varepsilon_r + \frac{\nu}{1-2\nu}\varepsilon_{kk} - \frac{1+\nu}{1-2\nu}\alpha\Delta T\right],\\
\sigma_\theta &= 2G\left[\varepsilon_\theta + \frac{\nu}{1-2\nu}\varepsilon_{kk} - \frac{1+\nu}{1-2\nu}\alpha\Delta T\right],\\
\sigma_z &= 2G\left[\varepsilon_z + \frac{\nu}{1-2\nu}\varepsilon_{kk} - \frac{1+\nu}{1-2\nu}\alpha\Delta T\right],\\
\tau_{r\theta} &= 2G\varepsilon_{r\theta}, \tau_{\theta z} = 2G\varepsilon_{\theta z}, \tau_{rz} = 2G\varepsilon_{rz},
\end{aligned}\right\} \tag{33}$$

5. Stress-Displacement Relations in Cylindrical Coordinates

Substituting the strain-displacement relations, Eqs. (13), into Eqs. (33) yields the following equations for the stress components in terms of the displacements:

$$\left.\begin{aligned}
\sigma_r &= 2G\left[\frac{\partial u_r}{\partial r} + \frac{\nu}{1-2\nu}\varepsilon_{kk} - \frac{1+\nu}{1-2\nu}\alpha\Delta T\right],\\
\sigma_\theta &= 2G\left[\frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\nu}{1-2\nu}\varepsilon_{kk} - \frac{1+\nu}{1-2\nu}\alpha\Delta T\right],\\
\sigma_z &= 2G\left[\frac{\partial w}{\partial z} + \frac{\nu}{1-2\nu}\varepsilon_{kk} - \frac{1+\nu}{1-2\nu}\alpha\Delta T\right],\\
\tau_{r\theta} &= G\left[\frac{1}{r}\frac{\partial u_r}{\partial r} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right],\\
\tau_{\theta z} &= G\left[\frac{\partial u_\theta}{\partial z} + \frac{1}{r}\frac{\partial w}{\partial \theta}\right],\\
\tau_{rz} &= G\left[\frac{\partial u_r}{\partial z} + \frac{\partial w}{\partial r}\right],
\end{aligned}\right\} \tag{34}$$

6. Equilibrium Equations in Terms of Displacements

The equilibrium equations for the stresses, Eqs. (22), by means of the stress-strain relations, Eqs. (25), can be expressed in terms of strain, and the strain components in turn can be expressed in terms of displacement. Such a formulation is of particular usefulness when the boundary conditions are given as prescribed displacements or rotations.

The equilibrium equations for the stresses in terms of the displacement components take the following form:

$$
\begin{aligned}
&(\lambda+\mu)\frac{\partial \varepsilon_{kk}}{\partial r} + \mu\nabla^2 u_r - (3\lambda+2\mu)\alpha\frac{\partial T}{\partial r} + F_r = 0\,,\\
&(\lambda+\mu)\frac{1}{r}\frac{\partial \varepsilon_{kk}}{\partial \theta} + \mu\nabla^2 u_\theta - (3\lambda+2\mu)\alpha\frac{\partial T}{\partial z} + F_\theta = 0\,,\\
&(\lambda+\mu)\frac{\partial \varepsilon_{kk}}{\partial z} + \mu\nabla^2 w - (3\lambda+2\mu)\alpha\frac{\partial T}{\partial z} + F_z = 0\,,
\end{aligned}
\tag{35}
$$

where ∇^2 is the three dimensional Laplacian operator in the cylindrical coordinate system r, θ, z:

$$
\nabla^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}\,.
$$

Substitution of the rotation components, Eqs. (14), yields the equilibrium equations for the stresses in the following form:

$$
\begin{aligned}
&(\lambda+2\mu)\frac{\partial \varepsilon_{kk}}{\partial r} - 2\mu\left(\frac{1}{r}\frac{\partial \omega_z}{\partial \theta} - \frac{\partial \omega_\theta}{\partial z}\right) - (3\lambda+2\mu)\alpha\frac{\partial T}{\partial r} + F_r = 0\,,\\
&(\lambda+2\mu)\frac{1}{r}\frac{\partial \varepsilon_{kk}}{\partial \theta} - 2\mu\left(\frac{1}{r}\frac{\partial \omega_r}{\partial z} - \frac{\partial \omega_z}{\partial r}\right) - (3\lambda+2\mu)\frac{\alpha}{r}\frac{\partial T}{\partial \theta} + F_\theta = 0\,,\\
&(\lambda+2\mu)\frac{\partial \varepsilon_{kk}}{\partial z} - \frac{2\mu}{r}\left(\frac{\partial (r\omega_\theta)}{\partial r} - \frac{\partial \omega_r}{\partial \theta}\right) - (3\lambda+2\mu)\alpha\frac{\partial T}{\partial z} + F_z = 0\,.
\end{aligned}
\tag{36}
$$

7. Mathematical Formulation of the Problem of Thermoelasticity

If the temperature distribution in a body s known, the problem of the elasticity consists in the determination of the following 15 functions (in cylindrical coordinates):

6 stress components: σ_r, σ_θ, σ_z, $\tau_{r\theta}$, $\tau_{\theta z}$, τ_{zr};

6 strain components: ε_r, ε_θ, ε_z, $\gamma_{r\theta}$, $\gamma_{\theta z}$, γ_{zr};

3 displacement components: u_r, u_θ, w,

so as to satisfy the following 15 equations throughout the body:

3 equilibrium equations: Eqs. (22)

6 stress-strain relations: Eqs. (24), or expressions in other form

6 strain-displacement relations: Eqs. (13),

as well as to satisfy the boundary conditions.

8. Bibliography

See e.g.:

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