Appendix—Elasticity Theory

A.1 EQUILIBRIUM CONDITIONS

Figure A.1 shows an interior region of a solid bounded by the closed surface S. If the region contained in S is not accelerating, Newton's law of motion requires that the components of the forces on the region in each of the coordinate directions must be zero. If body forces such as gravity are neglected, the only forces acting on the region are those exerted on the surface S by the material outside S.

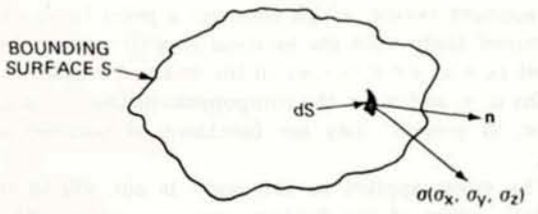

Fig. A.1 Region in a solid.

The forces on the surface S can be represented by a stress vector σ, which has components σ_x, σ_y, and σ_z in the three directions of a Cartesian coordinate system. The magnitude and direction of the stress vector vary with position on the surface S. Figure A.1 shows a small element of area dS of the surface S. The outward normal to the surface at that point is denoted by n. The stress vector at the location of dS will in general have a component along the normal, which represents a normal stress, and components tangent to the surface element, which are shear stresses.

The x-component of the force on the element of surface dS is σ_x dS. The x-component of the net x force on the entire region enclosed by the surface S is the surface integral of σ_x dS. At equilibrium this integral and its counterparts in the y- and z-directions must be zero, or

$$\int_S \sigma_x \, dS = 0$$

$$\int_S \sigma_y \, dS = 0 \qquad \text{(A.1)}$$

$$\int_S \sigma_z \, dS = 0$$

To express σ_x, σ_y, and σ_z in a more convenient manner, we consider the surface element dS in Fig. A.1 to be the oblique face of a very small tetrahedron, as shown in Fig. A.2. The three mutually perpendicular planes of the tetrahedron are each perpendicular to one of the coordinate axes. The components of σ can be expressed in terms of the unit normal n and the three stress vectors acting on the coordinate faces. The stress vectors on each coordinate plane are resolved into a normal stress component (e.g., σ_{xx}) and two shear components (e.g., σ_{xy} and σ_{xz}) that act tangentially to the coordinate plane.

The components of the stress vectors on the coordinate planes are written as σ_{ij}, where i refers to the coordinate plane in which the stress acts (e.g., σ_{xx}, σ_{xy}, and σ_{xz} all act on the plane perpendicular to the x axis) and j refers to the direction in which the stress component acts.

The area of each of the coordinate planes in Fig. A.2 is a projected area of the oblique face dS. The area over which σ_{xx}, σ_{xy}, and σ_{xz} act is $\mathbf{n} \cdot \mathbf{i}\, dS = n_x\, dS$, where i, j, and k are the unit vectors in the x, y, and z directions, respectively. The direction cosines of the surface normal are $n_x = \mathbf{n} \cdot \mathbf{i}$, $n_y = \mathbf{n} \cdot \mathbf{j}$, and $n_z = \mathbf{n} \cdot \mathbf{k}$.

The tetrahedron of Fig. A.2 is in mechanical equilibrium owing to the forces on its four faces, or the net force

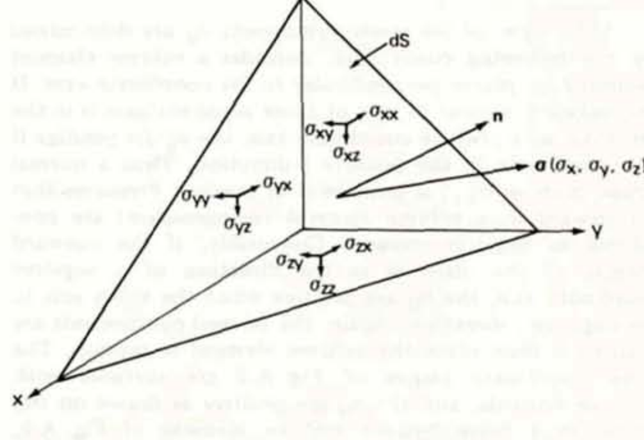

Fig. A.2 Diagram for relating surface forces to nine components of stress tensor.

in each coordinate direction is zero. According to the sign convention,* force components on the coordinate faces are positive if pointing toward the negative coordinate axis under consideration. The x-direction force balance is

$$\sigma_x \, dS - \sigma_{xx} n_x \, dS - \sigma_{yx} n_y \, dS - \sigma_{zx} n_z \, dS = 0$$

If dS is canceled and the last three terms are written as the scalar product of n and the vector whose components are σ_{xx}, σ_{yx}, and σ_{zx}, the preceding formula becomes

$$\sigma_x = \mathbf{n} \cdot (\sigma_{xx}\mathbf{i} + \sigma_{yx}\mathbf{j} + \sigma_{zx}\mathbf{k})$$

Similarly,

$$\sigma_y = \mathbf{n} \cdot (\sigma_{xy}\mathbf{i} + \sigma_{yy}\mathbf{j} + \sigma_{zy}\mathbf{k})$$

$$\sigma_z = \mathbf{n} \cdot (\sigma_{xz}\mathbf{i} + \sigma_{yz}\mathbf{j} + \sigma_{zz}\mathbf{k}) \qquad \text{(A.2)}$$

Equation A.2 relates the components of the stress vector σ to the unit normal describing the orientation of the surface and the nine components of the stresses acting on the three coordinate planes, usually denoted collectively as the *stress tensor:*

$$\sigma_{ij} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix} \qquad \text{(A.3)}$$

An important relation between the off-diagonal elements of the tensor σ_{ij} can be obtained by applying the condition that the angular acceleration of any element of volume is zero. This restriction leads to the relations

$$\sigma_{xy} = \sigma_{yx}$$

$$\sigma_{xz} = \sigma_{zx} \qquad \text{(A.4)}$$

$$\sigma_{zy} = \sigma_{yz}$$

Or, of the nine components of the stress tensor of Eq. A.3, only six are independent; the tensor is symmetric.

*The signs of the stress component σ_{ij} are determined by the following convention: consider a volume element bounded by planes perpendicular to the coordinate axes. If the outward normal of one of these plane surfaces is in the direction of a positive coordinate axis, the σ_{ij} are positive if the stress acts in the positive j direction. Thus a normal stress, such as σ_{xx}, is positive if in tension. Pressures that act inward in a volume element (compression) are considered as negative stresses. Conversely, if the outward normal of the plane is in the direction of a negative coordinate axis, the σ_{ij} are positive when the stress acts in the negative j direction. Again, the normal components are positive if they place the volume element in tension. The three coordinate planes of Fig. A.2 are surfaces with negative normals, and the σ_{ij} are positive as drawn on the figure. In a force balance on the element of Fig. A.2, therefore, the contribution to the total force in the positive x-direction due to the coordinate plane perpendicular to the x axis is $-\sigma_{xx}$ times the area of the plane.

Substitution of Eq. A.2 into Eq. A.1 yields equilibrium relations in terms of the stress tensor σ_{ij}. For the x-direction, the result is

$$\int_S \mathbf{n} \cdot (\sigma_{xx}\mathbf{i} + \sigma_{yx}\mathbf{j} + \sigma_{zx}\mathbf{k}) \, dS = 0$$

Applying the divergence theorem* to the surface integral in this equation yields the final form of the equilibrium relation:

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z} = 0 \qquad \text{(A.5)}$$

Similarly, for the y- and z-directions, the equilibrium conditions are

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{zy}}{\partial z} = 0 \qquad \text{(A.6)}$$

$$\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} = 0 \qquad \text{(A.7)}$$

A.2 DISPLACEMENTS

A body subjected to stress will become distorted or strained as a result. The state of strain is described by the displacement vector, which connects a point (x,y,z) in the unstrained body with the location to which this point has moved (x + u, y + v, z + w) in the strained condition. The lengths u, v, and w are the components of the displacement vector; in general, they are functions of position in the solid.

The stress applied to the body is not related to the absolute values of the displacement components. The fact that a body is made to undergo translation or rotation because of an applied force, for example, has nothing to do with the material properties of the solid. The property that is uniquely determined by the applied stress is the relative displacement of points in the solid. The solid-line rectangle in Fig. A.3 represents a plane rectangular element of dimensions δx and δy in an unstressed body (for simplicity, only two dimensions are considered). After stress has been applied, the rectangle is distorted into the dashed four-sided figure. The arrows connecting the corners of the unstrained and strained figures are the displacement vectors for these four points. The displacement vector of the point (x,y) in the lower left-hand corner has components u and v. Since the relative motion of adjacent points is small, the displacement components at other locations can be approximated by Taylor series expansions about the values at the point (x,y). Thus, the displacement components of the

*For a vector F defined over a region of volume V and surface S, the divergence theorem is

$$\int_S \mathbf{n} \cdot \mathbf{F} \, dS = \int_V \nabla \cdot \mathbf{F} \, dV$$

where, for Cartesian coordinates,

$$\nabla \cdot \mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

lower right-hand corner $(x + \delta x, y)$ are $u + (\partial u/\partial x)\,\delta x$ and $v + (\partial v/\partial x)\,\delta x$.

The aspect of the displacement field which is directly related to the applied stress and the material properties is the collection of derivatives of the displacement, $\partial u/\partial x$, $\partial v/\partial x, \ldots$, not the displacements u and v proper. In three dimensions there are nine spatial derivatives of the displacement-vector components which can be expressed as the strain tensor:

$$S_{ij} = \begin{pmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} & \dfrac{\partial u}{\partial z} \\[2ex] \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} & \dfrac{\partial v}{\partial z} \\[2ex] \dfrac{\partial w}{\partial x} & \dfrac{\partial w}{\partial y} & \dfrac{\partial w}{\partial z} \end{pmatrix} \qquad \text{(A.8)}$$

The response of the solid to an applied force is governed by the law that relates the stress tensor of Eq. A.3 and the strain tensor of Eq. A.8. However, the stress tensor contains only six independent components, whereas the strain tensor of Eq. A.8 contains nine independent quantities. Thus, the symmetric stress tensor cannot be directly related to the strain components as given in Eq. A.8. To circumvent this difficulty, we can split the strain tensor S_{ij} into two parts:

$$S_{ij} = \epsilon_{ij} + \omega_{ij} \qquad \text{(A.9)}$$

where ϵ_{ij} is the symmetric deformation tensor:

$$\epsilon_{ij} = \frac{1}{2} \begin{bmatrix} \left(\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial x}\right) & \left(\dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x}\right) & \left(\dfrac{\partial u}{\partial z} + \dfrac{\partial w}{\partial x}\right) \\[2ex] \left(\dfrac{\partial v}{\partial x} + \dfrac{\partial u}{\partial y}\right) & \left(\dfrac{\partial v}{\partial y} + \dfrac{\partial v}{\partial y}\right) & \left(\dfrac{\partial v}{\partial z} + \dfrac{\partial w}{\partial y}\right) \\[2ex] \left(\dfrac{\partial w}{\partial x} + \dfrac{\partial u}{\partial z}\right) & \left(\dfrac{\partial w}{\partial y} + \dfrac{\partial v}{\partial z}\right) & \left(\dfrac{\partial w}{\partial z} + \dfrac{\partial w}{\partial z}\right) \end{bmatrix} \qquad \text{(A.10)}$$

and ω_{ij} is the skew-symmetric rotation tensor:

$$\omega_{ij} = \frac{1}{2} \begin{bmatrix} 0 & \left(\dfrac{\partial u}{\partial y} - \dfrac{\partial v}{\partial x}\right) & \left(\dfrac{\partial u}{\partial z} - \dfrac{\partial w}{\partial x}\right) \\[2ex] \left(\dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial y}\right) & 0 & \left(\dfrac{\partial v}{\partial z} - \dfrac{\partial w}{\partial y}\right) \\[2ex] \left(\dfrac{\partial w}{\partial x} - \dfrac{\partial u}{\partial z}\right) & \left(\dfrac{\partial w}{\partial y} - \dfrac{\partial v}{\partial z}\right) & 0 \end{bmatrix} \qquad \text{(A.11)}$$

The physical meaning of the components of ϵ_{ij} and ω_{ij} in terms of the type of motion experienced by the body can be seen from Fig. A.3.

Consider the bottom edge of the rectangle. Originally its length was δx; under stress the corner (x,y) moves a distance u to the right, and the corner $(x + \delta x, y)$ moves a distance $u + (\partial u/\partial x)\,\delta x$ to the right. The strain is a fractional change in length, or

$$\frac{\text{Elongation in the x-direction}}{\text{Unit length}} = \frac{u + (\partial u/\partial x)\,\delta x - u}{\delta x} = \frac{\partial u}{\partial x}$$

This is the ϵ_{xx} component of the deformation tensor. Thus, the diagonal elements of Eq. A.10 represent the *elongations* or *normal strains* in the three coordinate directions.

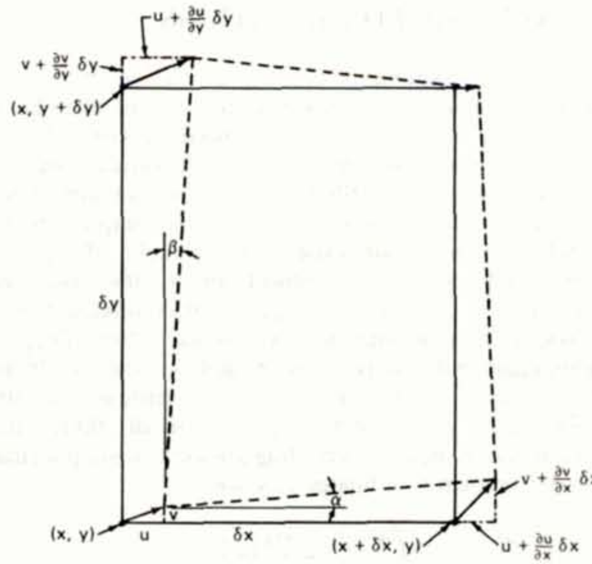

Fig. A.3 Deformation of a region of a solid (two-dimensional).

The angles α and β in Fig. A.3 represent the departure of the four-sided figure from its original rectangular shape. Since the strains are small, these angles are given by

$$\alpha \simeq \tan\alpha = \frac{v + (\partial v/\partial x)\,\delta x - v}{\delta x} = \frac{\partial v}{\partial x}$$

and

$$\beta \simeq \tan\beta = \frac{u + (\partial u/\partial y)\,\delta y - u}{\delta y} = \frac{\partial u}{\partial y}$$

The sum $\alpha + \beta$ represents the departure of the original angle from 90°, which is denoted as *shear strain*,

$$\text{Departure from } 90^\circ \text{ angle} = \alpha + \beta = \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

This sum is twice the ϵ_{xy} component of the deformation tensor. The other off-diagonal elements in Eq. A.10 represent the shear strains in the solid.

The angle of rotation of the plane figure in Fig. A.3 is the average of the angles α and β, taking positive rotation in the clockwise sense, or

$$\text{Rotation as a solid body} = \frac{1}{2}(\beta - \alpha) = \frac{1}{2}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right)$$

Thus the tensor ω_{ij} of Eq. A.11 represents pure rotation of the body.

Only the deformation tensor ϵ_{ij} is determined by the stress tensor and material properties. Hooke's law is an example of such a relation for elastic deformations.

A.3 COMPATIBILITY RELATIONS

Additional relationships between the deformation components ϵ_{ij} reflect the requirement that the medium be a continuum, or that the solid has not been cracked and that there are no discontinuities in the displacements. Mathematically, these *compatibility conditions* require that certain rather obvious relations among the components of the deformation tensor exist. For example, if ϵ_{xx} is differentiated twice with respect to y, the result is $(\partial^3 u)/(\partial x\, \partial y^2)$. Similarly, if ϵ_{yy} is differentiated twice with respect to x, we obtain $(\partial^3 v)/(\partial y\, \partial x^2)$. Now if ϵ_{xy} is differentiated with respect to x and y, the result is $(1/2)[(\partial^3 u)/(\partial x\, \partial y^2) + (\partial^2 v)/(\partial x^2\, \partial y)]$, which is one-half the sum of $\partial^2 \epsilon_{xx}/\partial y^2$ and $\partial^2 \epsilon_{yy}/\partial x^2$. In all, there are six compatibility equations relating the various components of ϵ_{ij}. In Cartesian coordinates they are

$$\frac{\partial^2 \epsilon_{xy}}{\partial x\, \partial y} = \frac{1}{2}\left(\frac{\partial^2 \epsilon_{xx}}{\partial y^2} + \frac{\partial^2 \epsilon_{yy}}{\partial x^2}\right)$$

$$\frac{\partial^2 \epsilon_{xx}}{\partial y\, \partial z} = \frac{\partial}{\partial x}\left(-\frac{\partial \epsilon_{yz}}{\partial x} + \frac{\partial \epsilon_{xz}}{\partial y} + \frac{\partial \epsilon_{xy}}{\partial z}\right)$$

$$\frac{\partial^2 \epsilon_{xz}}{\partial x\, \partial z} = \frac{1}{2}\left(\frac{\partial^2 \epsilon_{xx}}{\partial z^2} + \frac{\partial^2 \epsilon_{zz}}{\partial x^2}\right)$$

$$\frac{\partial^2 \epsilon_{yy}}{\partial x\, \partial z} = \frac{\partial}{\partial y}\left(-\frac{\partial \epsilon_{xz}}{\partial y} + \frac{\partial \epsilon_{xy}}{\partial z} + \frac{\partial \epsilon_{yz}}{\partial x}\right)$$

$$\frac{\partial^2 \epsilon_{yz}}{\partial y\, \partial z} = \frac{1}{2}\left(\frac{\partial^2 \epsilon_{yy}}{\partial z^2} + \frac{\partial^2 \epsilon_{zz}}{\partial y^2}\right)$$

$$\frac{\partial^2 \epsilon_{zz}}{\partial x\, \partial y} = \frac{\partial}{\partial z}\left(-\frac{\partial \epsilon_{xy}}{\partial z} + \frac{\partial \epsilon_{yz}}{\partial x} + \frac{\partial \epsilon_{xz}}{\partial y}\right) \tag{A.12}$$

A.4 STRESS–STRAIN RELATIONS

In the absence of plastic deformation, creep, or temperature changes, the stress–strain relation is given by the generalized form of Hooke's law. For the elastic solid, the six components of the stress tensor are related to the six components of the deformation tensor by the linear equations:

$$\sigma_{xx} = c_{11}\epsilon_{xx} + c_{12}\epsilon_{yy} + c_{13}\epsilon_{zz} + c_{14}\epsilon_{yz} + c_{15}\epsilon_{zx} + c_{16}\epsilon_{xy}$$

$$\sigma_{yy} = c_{21}\epsilon_{xx} + c_{22}\epsilon_{yy} + c_{23}\epsilon_{zz} + c_{24}\epsilon_{yz} + c_{25}\epsilon_{zx} + c_{26}\epsilon_{xy}$$

$$\sigma_{zz} = c_{31}\epsilon_{xx} + c_{32}\epsilon_{yy} + c_{33}\epsilon_{zz} + c_{34}\epsilon_{yz} + c_{35}\epsilon_{zx} + c_{36}\epsilon_{xy}$$

$$\sigma_{yz} = c_{41}\epsilon_{xx} + c_{42}\epsilon_{yy} + c_{43}\epsilon_{zz} + c_{44}\epsilon_{yz} + c_{45}\epsilon_{zx} + c_{46}\epsilon_{xy}$$

$$\sigma_{zx} = c_{51}\epsilon_{xx} + c_{52}\epsilon_{yy} + c_{53}\epsilon_{zz} + c_{54}\epsilon_{yz} + c_{55}\epsilon_{zx} + c_{56}\epsilon_{xy}$$

$$\sigma_{xy} = c_{61}\epsilon_{xx} + c_{62}\epsilon_{yy} + c_{63}\epsilon_{zz} + c_{64}\epsilon_{yz} + c_{65}\epsilon_{zx} + c_{66}\epsilon_{xy} \tag{A.13}$$

The c_{ij} values are the *elastic moduli* of the medium. Not all 36 of the coefficients in Eq. A.13 are independent. Because the tensors σ_{ij} and ϵ_{ij} are symmetric, $c_{ij} = c_{ji}$, which reduces the number of elastic constants to 21. This number is reduced still further according to the symmetry of the crystal structure of the solid; the greater the symmetry, the fewer the constants. For crystals of the cubic system, there are only three elastic constants. Finally, for materials that are macroscopically isotropic (either because the substance is noncrystalline or because the material is in polycrystalline form), only two constants remain. These two elastic constants are called *Lamé coefficients* λ and μ. They determine the stress–strain relation by

$$\sigma_{ii} = 2\mu\epsilon_{ii} + \lambda\delta \tag{A.14}$$

and

$$\sigma_{ij} = 2\mu\epsilon_{ij} \qquad (i \neq j) \tag{A.15}$$

Here δ is the volume dilatation, or the fractional change in volume:

$$\delta = \epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz} \tag{A.16}$$

The elastic constants are usually expressed in terms of Young's modulus E, the shear modulus G, and Poisson's ratio ν, instead of the Lamé coefficients. The relations between the conventional elastic moduli and the Lamé coefficients are

$$E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu} \tag{A.17}$$

$$G = \mu \tag{A.18}$$

$$\nu = \frac{\lambda}{2(\lambda + \mu)} \tag{A.19}$$

The values E, G, and ν are not independent but are related by

$$G = \frac{E}{2(1 + \nu)} \tag{A.20}$$

Using Eqs. A.17 to A.19 in Eqs. A.14 and A.15 and solving for the strains yields

$$\epsilon_{xx} = \frac{1}{E}[\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz})]$$

$$\epsilon_{yy} = \frac{1}{E}[\sigma_{yy} - \nu(\sigma_{xx} + \sigma_{zz})] \quad (A.21)$$

$$\epsilon_{zz} = \frac{1}{E}[\sigma_{zz} - \nu(\sigma_{xx} + \sigma_{yy})]$$

$$\epsilon_{xy} = \frac{1}{2G}\sigma_{xy} \qquad \epsilon_{xz} = \frac{1}{2G}\sigma_{xz} \qquad \epsilon_{yz} = \frac{1}{2G}\sigma_{yz} \quad (A.22)$$

In addition to the elongations caused by applied stresses, a change in temperature produces normal strains (but not shear strains) given by

$$(\epsilon_{ii})_{\text{thermal}} = \alpha\,\Delta T \quad (A.23)$$

where α is the coefficient of linear thermal expansion and ΔT is the temperature rise with respect to a reference temperature. The thermal component given by Eq. A.23 is added to each of the normal strains given by Eq. A.21 to produce the total strain.

Effects other than applied stress or temperature change can contribute to the strain. In analyses of the performance of reactor fuel elements, for example, the elongations of Eq. A.21 are supplemented by contributions due to creep and fission-product swelling. Like the thermal component of the strain, these effects are accommodated into the stress–strain relations by adding appropriate terms to the right-hand sides of Eq. A.21. Relations such as Eqs. A.21 and A.22 to which other sources of displacement have been appended are known as *constitutive relations.*

A.5 ELASTIC STRAIN ENERGY

The strain of a solid as a result of applied stresses means that work has been done on the material. This work is stored as internal energy, or *elastic strain energy*, in the medium.

The strain energy can best be illustrated by considering the one-dimensional analog of the solid, namely, the elastic string. If sufficient force is applied to a string to extend its length from x_0 to x_f, the work done in the process is

$$W = k\int_{x_0}^{x_f}(x - x_0)dx = \frac{1}{2}k(x_f - x_0)^2$$

Here k is the Hooke's law constant of the string. The force on the string in the final state is

$$F = k(x_F - x_0)$$

The work done can also be written as

$$W = \frac{1}{2}F(x_F - x_0)$$

The elastic energy E_{el} stored in a unit length of string is W/x_0. The strain ϵ of the string is $(x_f - x_0)/x_0$. Dividing the preceding formula by the initial length yields

$$E_{el} = \frac{1}{2}F\epsilon \quad (A.24)$$

In a three-dimensional elastic medium, the single force F is replaced by the components of the stress tensor, and the strain is represented by the deformation tensor. The strain energy per unit volume is

$$E_{el} = \frac{1}{2}(\sigma_{xx}\epsilon_{xx} + \sigma_{yy}\epsilon_{yy} + \sigma_{zz}\epsilon_{zz} + 2\sigma_{xy}\epsilon_{xy} + 2\sigma_{xz}\epsilon_{xz} + 2\sigma_{yz}\epsilon_{yz}) \quad (A.25)$$

The elastic-energy density can also be written in terms of the stresses alone by substituting Eq. A.21 into Eq. A.25:

$$E_{el} = \frac{1}{2E}(\sigma_{xx}^2 + \sigma_{yy}^2 + \sigma_{zz}^2) - \frac{\nu}{E}(\sigma_{xx}\sigma_{yy} + \sigma_{xx}\sigma_{zz} + \sigma_{yy}\sigma_{zz}) + \frac{1}{2G}(\sigma_{xy}^2 + \sigma_{xz}^2 + \sigma_{yz}^2) \quad (A.26)$$

For the case of a simple hydrostatic stress system ($\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = \sigma$ and $\sigma_{xy} = \sigma_{xz} = \sigma_{yz} = 0$), Eq. A.26 reduces to

$$E_{el} = \frac{\sigma^2}{2K} \quad (A.27)$$

where

$$K = \frac{E}{3(1 - 2\nu)} \quad (A.28)$$

is the bulk modulus, which is the reciprocal of the coefficient of compressibility (see problem 1.5, Chap. 1).

A.6 CYLINDRICAL COORDINATES

The behavior of a solid under applied stresses is determined by simultaneous application of the equilibrium conditions (Sec. A.1), the compatibility conditions (Sec. A.3), and a stress–strain relation (Sec. A.4). Strains and displacements are related by the components of the deformation tensor (Sec. A.2).

The analysis up to this point has been conducted in terms of Cartesian coordinates. However, many important problems (e.g., the stresses around a dislocation or in a reactor fuel element) are more conveniently treated in cylindrical coordinates. For this purpose the four relations listed in the preceding paragraph must be transformed from rectangular to cylindrical coordinates.

Transformation of the stress–strain relation requires only the replacement of x, y, and z in Eqs. A.21 and A.22 by the radial, azimuthal, and axial coordinates r, θ, and z.

Since the number of relevant strain components is considerably reduced when simple shapes are treated, the compatibility relations for cylindrical coordinates are best determined from the set of strain components peculiar to the problem at hand. The method of generating compatibility relations for cylindrical coordinates is analogous to that used in Sec. A.3 for rectangular coordinates.

Transformation of the equilibrium relations and the components of the deformation tensor is straightforward but tedious. The results are

Equilibrium conditions:

$$\frac{1}{r}\frac{\partial}{\partial r}(r\sigma_{rr}) + \frac{1}{r}\frac{\partial \sigma_{r\theta}}{\partial \theta} - \frac{1}{r}\sigma_{\theta\theta} + \frac{\partial \sigma_{rz}}{\partial z} = 0 \quad \text{(A.29)}$$

$$\frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \sigma_{r\theta}}{\partial r} + \frac{2}{r}\sigma_{r\theta} + \frac{\partial \sigma_{\theta z}}{\partial z} = 0 \quad \text{(A.30)}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(r\sigma_{rz}) + \frac{1}{r}\frac{\partial \sigma_{\theta z}}{\partial \theta} + \frac{\partial \sigma_{zz}}{\partial z} = 0 \quad \text{(A.31)}$$

Components of the deformation tensor:

$$\epsilon_{rr} = \frac{\partial u_r}{\partial r} \quad \epsilon_{\theta\theta} = \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} \quad \epsilon_{zz} = \frac{\partial u_z}{\partial z} \quad \text{(A.32)}$$

$$\epsilon_{r\theta} = \epsilon_{\theta r} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right)$$

$$\epsilon_{rz} = \epsilon_{zr} = \frac{1}{2}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right) \quad \text{(A.33)}$$

$$\epsilon_{z\theta} = \epsilon_{\theta z} = \frac{1}{2}\left(\frac{\partial u_\theta}{\partial z} + \frac{1}{r}\frac{\partial u_z}{\partial \theta}\right)$$

Stress components and displacements in cylindrical coordinates are shown in Fig. A.4. The θ-direction is orthogonal to the radial and axial directions.

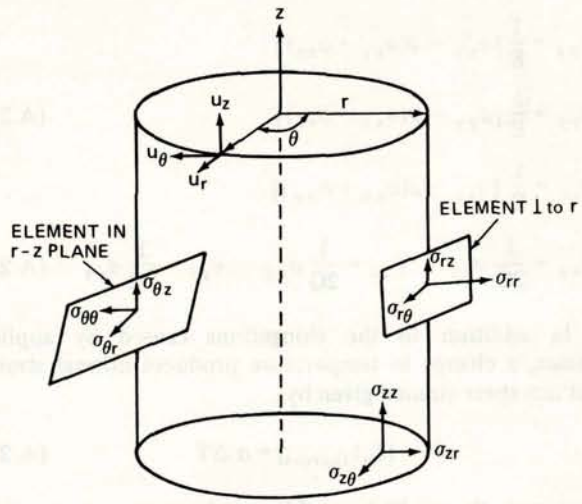

Fig. A.4 Stress components and displacements in the cylindrical coordinate system.

A.7 NOMENCLATURE

c_{ij} = coefficients of generalized Hooke's law (elastic moduli)
E = Young's modulus
E_{el} = elastic strain energy per unit volume
F = force
G = shear modulus
i,j,k = unit vectors
K = bulk modulus
n = outward normal to surface
r,θ,z = cylindrical coordinates
S_{ij} = strain tensor
T = temperature
u,v,w = components of displacement vector
W = work
x,y,z = Cartesian coordinates

Greek Letters

α = coefficient of linear thermal expansion
δ = volume dilation
ϵ_{ij} = symmetric deformation tensor (strain components)
λ,μ = Lamé constants for an isotropic solid
ν = Poisson's ratio
σ_{ij} = stress tensor (stress components)
ω_{ij} = skew-symmetric rotation tensor

A.8 ADDITIONAL READING

1. J. Weertman and J. R. Weertman, *Elementary Dislocation Theory*, Chap. 2, The Macmillan Company, New York, 1964.
2. Z. Zudans, T. C. Yen, and W. H. Steigelmann, *Thermal Stress Techniques in the Nuclear Industry*, Chap. 3, American Elsevier Publishing Company, Inc., New York, 1965.