NOTES L.10

ARTICLE A-2000

ANALYSIS OF CYLINDRICAL SHELLS

A-2100 INTRODUCTION

A-2110 SCOPE

(a) In this Article formulas are given for stress and deformations in cylindrical shells subjected to internal pressure only. For cylindrical shells subjected to external pressure, see NB-3133.3

(b) Formulas are also given for bending analysis for uniformly distributed edge loads.

A-2120 NOMENCLATURE AND SIGN CONVENTION

The symbols and sign convention adopted in this Article for the analysis of cylindrical shells are defined as follows:

(1) $D = Et^3/12(1-\nu^2)$, pound-inch
(2) $E =$ modulus of elasticity, psi
(3) $L =$ length of cylinder, in.—used as subscript to denote evaluation of a quantity at end of cylinder removed from reference end
(4) $M =$ longitudinal bending moment per unit length of circumference, inch-pounds per inch
(5) $o =$ used as subscript to denote evaluation of a quantity at reference end of cylinder, $x=0$
(6) $p =$ internal pressure, psi
(7) $Q =$ radial shearing forces per unit length of circumference, pounds per inch
(8) $R =$ inside radius, in.
(9) $S =$ stress intensity, psi
(10) $t =$ thickness of cylinder, in.
(11) $w =$ radial displacement of cylinder wall, in.
(12) $x =$ axial distance measured from the reference end of cylinder, in.
(13) $Y =$ ratio of outside radius to inside radius
(14) $Z =$ ratio of outside radius to an intermediate radius
(15) $\beta = [3(1-\nu^2)/(R+t/2)^2t^2]^{1/4}$, 1/in.
(16) $\theta = dw/dx =$ rotation of cylinder wall, radians
(17) $\nu =$ Poisson's ratio
(18) $\sigma_t =$ tangential (circumferential) stress component, psi
(19) $\sigma_l =$ longitudinal (meridional) stress component, psi
(20) $\sigma_r =$ radial stress component, psi
(21) $F_{11}(\beta x) = (\cosh \beta x \sin \beta x - \sinh \beta x \cos \beta x)/2$
(22) $F_{12}(\beta x) = \sinh \beta x \sin \beta x$
(23) $F_{13}(\beta x) = (\cosh \beta x \sin \beta x + \sinh \beta x \cos \beta x)/2$

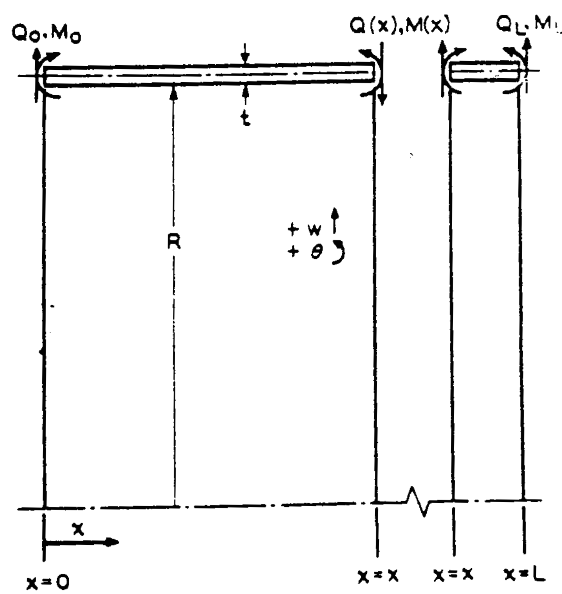

FIG. A-2120-1

The sign convention arbitrarily chosen for the analysis of cylindrical shells in this Article is as indicated in Fig. A-2120-1. Positive directions assumed for pertinent quantities are indicated.

REFERENCES

[1] B. F. Langer, "Design Values for Thermal Stress in Ductile Materials," *Welding J. Res. Suppl.*, Vol. 37, September 1958, p. 411-s.

[2] B. F. Langer, "Design of Pressure Vessels for Low-Cycle Fatigue," *J. Basic Eng.*, Vol. 84, No. 3, September 1962, p. 389.

[3] R. E. Peterson, "Stress Concentration Design Factors," John Wiley and Sons, Inc., New York, N.Y., 1953.

[4] R. B. Heywood, "Designing by Photoelasticity," Chapman & Hall, Ltd., London, 1952.

[5] W. N. Findley, P. N. Mathur, E. Szczepanski and A. O. Temel, "Energy Versus Stress Theories for Combined Stress—A Fatigue Experiment Using a Rotating Disk," *J. Basic Eng.*, Vol. 83, No. 1, March 1961.

[6] E. E. Baldwin, G. J. Sokol, and L. F. Coffin, Jr., "Cyclic Strain Fatigue Studies on AISI Type 347 Stainless Steel," *ASTM Proc.*, Vol. 57, 1957, p. 567.

[7] L. F. Kooistra and M. M. Lemcoe, "Low Cycle Fatigue Research on Full-Size Pressure Vessels," *Welding Res. Suppl.*, July 1962, p. 297-s.

[8] G. Welter and J. Dubuc, "Fatigue Resistance of Simulated Nozzles in Model Pressure Vessels of T-1 Steel," *Welding J. Res. Suppl.*, August 1962, p. 368-s.

[9] C. W. Lawton, "High-temperature Low-cycle Fatigue: A Summary of Industry and Code Work," *Experimental Mechanics*, June, 1968, p. 264.

[10] A. L. Snow, and B. F. Langer, "Low-cycle Fatigue of Large Diameter Bolts," *Journal of Engineering for Industry*, Vol. 89, Ser. B, No. 1, Feb., 1967, p. 53.

[11] R. J. Shield and D. C. Drucker, "Design of Thin-walled Torispherical and Toriconical Pressure Vessel Heads," *Journal of Applied Mechanics*, Vol. 28, Ser. E, June, 1962, p. 262.

[12] D. R. Miller, "Thermal-stress Ratchet Mechanism in Pressure Vessels," *ASME Transactions*, Vol. 81, Ser. D, No. 2, 1959.

(24) $F_{11}(\beta x) = \cosh \beta x \cos \beta x$

(25) $f_1(\beta x) = e^{-\beta x} \cos \beta x$

(26) $f_2(\beta x) = e^{-\beta x}(\cos \beta x - \sin \beta x)$

(27) $f_3(\beta x) = e^{-\beta x}(\cos \beta x + \sin \beta x)$

(28) $f_4(\beta x) = e^{-\beta x} \sin \beta x$

(29) $B_{11} = B_{11}(\beta L) = (\sinh 2\beta L - \sin 2\beta L)/2(\sinh^2 \beta L - \sin^2 \beta L)$

(30) $B_{12} = B_{12}(\beta L) = (\cosh 2\beta L - \cos 2\beta L)/2(\sinh^2 \beta L - \sin^2 \beta L)$

(31) $B_{22} = B_{22}(\beta L) = (\sinh 2\beta L + \sin 2\beta L)/(\sinh^2 \beta L - \sin^2 \beta L)$

(32) $G_{11} = G_{11}(\beta L) = -(\cosh \beta L \sin \beta L - \sinh \beta L \cos BL)/(\sinh^2 \beta L - \sin^2 \beta L)$

(33) $G_{12} = G_{12}(\beta L) = -2 \sinh \beta L \sin \beta L/(\sinh^2 \beta L - \sin^2 \beta L)$

(34) $G_{22} = G_{22}(\beta L) = -2 (\cosh \beta L \sin \beta L + \sinh \beta L \cos \beta L)/(\sinh^2 \beta L - \sin^2 \beta L)$.

A-2200 STRESS INTENSITIES, DISPLACEMENTS, BENDING MOMENTS AND LIMITING VALUES

A-2210 PRINCIPAL STRESSES AND STRESS INTENSITIES DUE TO INTERNAL PRESSURE

A-2211 Loading Effects Considered

In this subarticle formulas are given for principal stresses and stress intensities resulting from uniformly distributed internal pressure in cylindrical shells. The effects of discontinuities in geometry and loading are not included and should be evaluated independently. The stresses resulting from all effects must be combined by superposition.

A-2212 Principal Stresses

The principal stresses developed at any point in the wall of a cylindrical shell due to internal pressure are given by the formulas:

$$\sigma_1 = \sigma_t = p(1 + Z^2)/(Y^2 - 1) \qquad \text{(a)}$$

$$\sigma_2 = \sigma_l = p/(Y^2 - 1) \qquad \text{(b)}$$

$$\sigma_3 = \sigma_r = p(1 - Z^2)/(Y^2 - 1) \qquad \text{(c)}$$

A-2220 STRESS INTENSITIES

A-2221 General Primary-Membrane Stress Intensity

The general primary-membrane stress intensity developed in a cylindrical shell as a result of internal pressure is given by the formula:

$$S = \frac{pR}{t} + \frac{p}{2} \qquad \text{(a)}$$

A-2222 Maximum Value of Primary-Plus-Secondary Stress Intensity

The maximum value of the primary-plus-secondary stress intensity in a cylindrical shell as a result of internal pressure occurs at the inside surface and is given by the formula:

$$S = 2pY^2/(Y^2 - 1) \qquad \text{(b)}$$

A-2223 Values of Radial Stress Used

Note that in evaluating the general primary-membrane stress intensity, the average value of the radial stress has been taken as $(-p/2)$. This has been done to obtain a result consistent with burst-pressure analyses. On the other hand, the radial stress value used in A-2222 is $(-p)$, the value at the inner surface, since the purpose of that quantity is to control local behavior.

A-2230 BENDING ANALYSIS FOR UNIFORMLY DISTRIBUTED EDGE LOADS

A-2231 Behavior of Shells Subjected to Bending Moments

The formulas in this subsubarticle describe the behavior of cylindrical shells when subjected to the action of bending moments, M inch-pounds per inch of circumference, and radial shearing forces, Q pounds per inch of circumference, uniformly distributed at the edges and acting at the mean radius of the shell. The behavior of shells due to all other loadings must be evaluated independently and combined by superposition.

A-2240 DISPLACEMENTS, BENDING MOMENTS AND SHEARING FORCES IN TERMS OF CONDITIONS AT REFERENCE EDGE, $x = o$

A-2241 Formulas for Conditions of Any Axial Location

The radial displacement, $w(x)$, the angular displacement or rotation, $\theta(x)$, the bending moments, $M(x)$, and the radial shearing forces, $Q(x)$, at any axial location of the cylinder are given by the following equations in terms of w_o, θ_o, M_o and Q_o.

$$w(x) = (Q_o/2\beta^3 D)F_{11}(\beta x) + (M_o/2\beta^2 D)F_{12}(\beta x) + (\theta_o/\beta)F_{13}(\beta x) + w_o F_{14}(\beta x) \qquad \text{(a)}$$

$$\theta(x)/\beta = (Q_o/2\beta^3 D)F_{12}(\beta x) + 2(M_o/2\beta^2 D)F_{13}(\beta x) + (\theta_o/\beta)F_{14}(\beta x) - 2w_o F_{11}(\beta x) \qquad \text{(b)}$$

$$M(x)/2\beta^2 D = (Q_o/2\beta^3 D)F_{13}(\beta x) + (M_o/2\beta^2 D)F_{14}(\beta x) - (\theta_o/\beta)F_{11}(\beta x) - w_o F_{12}(\beta x) \qquad \text{(c)}$$

$$Q(x)/2\beta^3 D = (Q_o/2\beta^3 D)F_{14}(\beta x) - 2(M_o/2\beta^2 D)F_{11}(\beta x) - (\theta_o/\beta)F_{12}(\beta x) - 2w_o F_{13}(\beta x) \qquad \text{(d)}$$

A-2242 Formulas When Cylinder Length $\geq 3/\beta$

In the case of cylinders of sufficient length, the equations in A-2241 above reduce to those given below. These equations may be used for cylinders characterized by lengths not less than $3/\beta$. The combined effects of loadings at the two edges may be evaluated by applying the equations to the loadings at each edge, separately, and superposing the results.

$$w(x) = (Q_o/2\beta^3 D)f_1(\beta x) + (M_o/2\beta^2 D)f_2(\beta x) \qquad \text{(a)}$$

$$\theta(x)/\beta = -(Q_o/2\beta^3 D)f_2(\beta x) - 2(M_o/2\beta^2 D)f_1(\beta x) \qquad \text{(b)}$$

$$M(x)/2\beta^2 D = (Q_o/2\beta^3 D)f_4(\beta x) + (M_o/2\beta_2 D)f_3(\beta x) \qquad \text{(c)}$$

$$Q(x)/2\beta^3 D = (Q_o/2\beta^3 D)f_2(\beta x) - 2(M_o/2\beta^2 D)f_4(\beta x) \qquad \text{(d)}$$

A-2243 Edge-Displacements and Rotations in Terms of Edge Loads

The radial displacements, w_o and w_L, and rotations, θ_o and $-\theta_L$, developed at the edges of a cylindrical shell sustaining the action of edge loads Q_o, M_o, Q_L and M_L are given by the following formulas:

$$w_o = (B_{11}/2\beta^3 D)Q_o + (B_{12}/2\beta^2 D)M_o + (G_{11}/2\beta^3 D)Q_L + (G_{12}/2\beta^2 D)M_L \qquad \text{(a)}$$

$$-\theta_o = (B_{12}/2\beta^2 D)Q_o + (B_{22}/2\beta D)M_o + (G_{12}/2\beta^2 D)Q_L + (G_{22}/2\beta D)M_L \qquad \text{(b)}$$

$$w_L = (G_{11}/2\beta^3 D)Q_o + (G_{12}/2\beta^2 D)M_o + (B_{11}/2\beta^3 D)Q_L + (B_{12}/2\beta^2 D)M_L \qquad \text{(c)}$$

$$\theta_L = (G_{12}/2\beta^2 D)Q_o + (G_{22}/2\beta D)M_o + (B_{12}/2\beta^2 D)Q_L + (B_{22}/2\beta D)M_L \qquad \text{(d)}$$

A-2250 LIMITING VALUE OF FUNCTIONS

A-2251 General Limiting Values of Influence Functions

The influence functions, B's and G's, appearing in the formulas in A-2243 above, rapidly approach limiting values as the length, L, of the cylinder increases. The limiting values are

$$B_{11} = B_{12} = 1,\ B_{22} = 2,$$
$$G_{11} = G_{12} = G_{22} = 0.$$

(a) Thus, for cylindrical shells of sufficient length, the loading conditions prescribed at one edge do not influence the displacements at the other edge.

(b) In the case of cylindrical shells characterized by lengths not less than $3/\beta$, the influence functions, B's and G's are sufficiently close to the limiting values that the limiting values may be used in the formulas in A-2243 above, without significant error.

A-2252 Limiting Values of Influence Functions for Short Cylinders

In the case of sufficiently short cylinders, the influence functions, B's and G's, appearing in the formulas in A-2243 above, are, to a first approximation, given by the following expressions:

$$B_{11} = 2/\beta L \qquad G_{11} = -1/\beta L$$
$$B_{12} = 3/(\beta L)^2 \qquad G_{12} = -3/(\beta L)^2$$
$$B_{22} = 6/(\beta L)^3 \qquad G_{22} = -6/(\beta L)^3$$

Introducing these expressions for the influence functions, B's and G's, into the formulas in A-2243 above, yields expressions identical to those obtained by the application of ring theory. Accordingly, the resultant expressions are subject to all of the limitations inherent in the ring theory, including the limitations due to the assumption that the entire cross-sectional area of the ring, $t \times L$, rotates about its centroid without distortion. Nevertheless, in the analysis of very short cylindrical shells characterized by lengths not greater than $1/2\beta$, the expressions may be used without introducing significant error.

A-2260 PRINCIPAL STRESSES DUE TO BENDING

The principal stresses developed at the surfaces of a cylindrical shell at any axial location, x, due to uniformly distributed edge loads (see Fig. A-2120-1) are given by the formulas:

$$\sigma_1 = \sigma_\phi(x) = Ew(x)/(R + t/2) \pm 6\nu M(x)/t^2 \qquad \text{(a)}$$

$$\sigma_2 = \sigma_x(x) = \pm 6M(x)/t^2 \qquad \text{(b)}$$

$$\sigma_3 = \sigma_r \quad = 0 \qquad \text{(c)}$$

In these formulas where terms are preceded by a double sign, $\pm$, the upper sign refers to the inside surface of the cylinder and the lower sign refers to the outside surface.