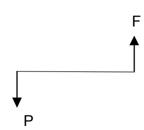
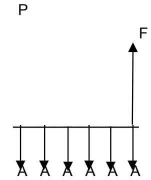
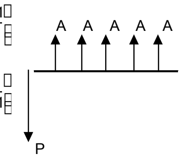
The Time Value of Money (contd.)

February 11, 2004

Time Value Equivalence Factors
(Discrete compounding, discrete payments)

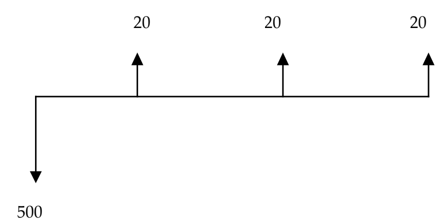
Factor Name	Factor Notation	Formula	Cash Flow Diagram
Future worth factor (compound amount factor)	(F/P, i, N)	$F=P(1+i)^N$	
Present worth factor	(P/F, i, N)	$P=F(1+i)^{-N}$	
Uniform series compound amount factor (aka future-worth-of-an-annuity factor)	(F/A, i, N)	$F = A\left[\frac{(1+i)^N - 1}{i}\right]$	
Sinking fund factor	(A/F, i, N)	$A = F\left[\frac{i}{(1+i)^N - 1}\right]$	
Present worth of an annuity factor	(P/A, i, N)	$P = A\left[\frac{(1+i)^N - 1}{i(1+i)^N}\right]$	
Capital recovery factor	(A/P, i, N)	$A = P\left[\frac{i(1+i)^N}{(1+i)^N - 1}\right]$	

Homework problem 2.1

A typical bank offers you a Visa card that charges interest on unpaid balances at a 1.5% per month compounded monthly. This means that the nominal interest rate (annual percentage) rate for this account is A and the effective annual interest rate is B. Suppose your beginning balance was $500 and you make only the required minimum monthly payment (payable at the end of each month) of $20 for the next 3 months. If you made no new purchases with this card during this period, your unpaid balance will be C at the end of the 3 months. What are the values of A,B, and C?

A = (1.5%)(12) = 18%
B = $(1+0.015)^{12}$-1 = 19.56%

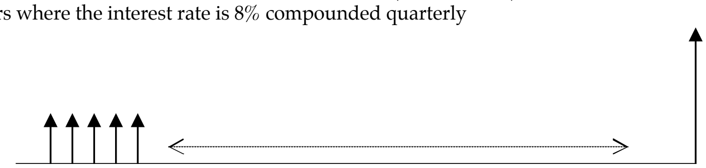
To do this problem we construct a cash-flow diagram



C = -500 (F/P, 1.5%, 3) + 20 (F/A,1.5%,3) = -$461.93

Homework problem 2.4

Suppose that $1,000 is placed in a bank account at the end of each quarter over the next 10 years. Determine the total accumulated value (future worth) at the end of the 10 years where the interest rate is 8% compounded quarterly



$$F = A\left[\frac{(1+i_q)^N - 1}{i_q}\right] = \$1{,}000\left[\frac{\left(1+\frac{0.08}{4}\right)^{40} - 1}{\frac{0.08}{4}}\right] = \$60{,}402$$

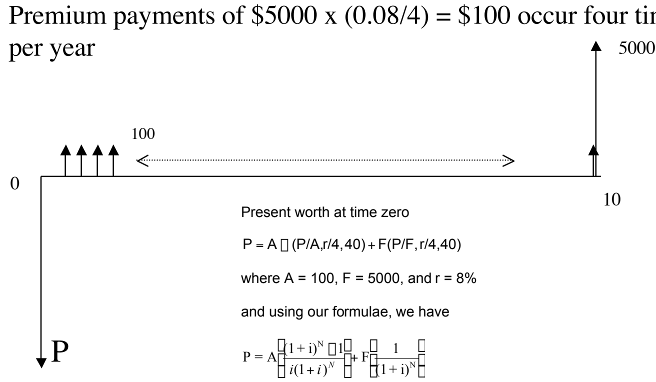
Example -- Valuation of Bonds

- Bonds are sold by organizations to raise money
- The bond represents a debt that the organization owes to the bondholder (not a share of ownership)
- Bonds typically bear interest semi-annually or quarterly, and are redeemable for a specified maturity value (also known as the face value) at a given maturity date.
- Interest is paid in the form of regular 'premiums'. The flow of premiums constitutes an annuity, A, where

  A = (face value) x (bond rate)
- Bonds can be bought and sold on the open market before they reach maturity
- The value (price) of a bond at a given point in time is equal to the present worth of the remaining premium payments plus the present worth of the redemption payment (i.e., the face value)

Example -- Valuation of Bonds (contd.)

- Consider a 10-year U.S. treasury bond with a face value of \$5000 and a bond rate of 8 percent, payable quarterly:
  - Premium payments of \$5000 x (0.08/4) = \$100 occur four times per year



Present worth at time zero

$P = A (P/A, r/4, 40) + F(P/F, r/4, 40)$

where A = 100, F = 5000, and r = 8%

and using our formulae, we have

$$P = A\left[\frac{(1+i)^N - 1}{i(1+i)^N}\right] + F\left[\frac{1}{(1+i)^N}\right]$$

and since

$A = Fi$

we have

$P = F$

Another example

- See:
  - What Exactly Is a Bond?
- What exactly is the mistake in this applet?

General bond valuation problem

Let:

Z = face, or par, value

C = redemption or disposal price (usually equal to Z)

r = bond rate (nominal interest) per period ("coupon")

N = number of periods before redemption

i = "yield to maturity" of bond = total return on bond at a given purchase price

V_N= value (price) of the bond N interest periods before redemption

The price of the bond is equal to the present worth of the future stream of payments paid by the borrower to the bondholder. This consists of (1) the series of periodic interest payments, and (2) the redemption value of the bond at retirement.

$$V_N = C\,(P/F, i\%, N) + rZ\,(P/A, i\%, N)$$

Note the difference between the coupon rate, r, and the yield rate i. The coupon rate r is fixed for a given bond, but the yield i depends on the bond purchase price. The desired yield is determined by the rate of interest in the economy. If the 'general' interest rate goes up, the yield required by bond investors will also go up, and hence the bond price today will decline.

Example

Find the current price of a 10-year bond paying 6% per year (payable semi-annually) that is redeemable at par value, if the purchaser requires an effective annual yield of 10% per year. The par value of the bond is $1000.

N = 10 x 2 = 20 periods

r = 6%/2 = 3% per period

Yield i per semi-annual period given by $(1+i)^2 = 1+0.1 = 1.1$

==> i = 0.049 = 4.9% per semi-annual period

C = Z = $1000

V_N = $1000 (P/F, 4.9%, 20) + $1000x0.03(P/A, 4.9%, 20)

= 384.1 + 377.06 = $761.16

Homework problem

Suppose you have the choice of investing in (1) a zero-coupon bond that costs $513.60 today, pays nothing during its life, and then pays $1,000 after 5 years or (2) a municipal bond that costs $1,000 today, pays $67 semiannually, and matures at the end of the 5 years. Which bond would provide the higher yield to maturity (or return on your investment).

Draw the cash flow diagram for each option:

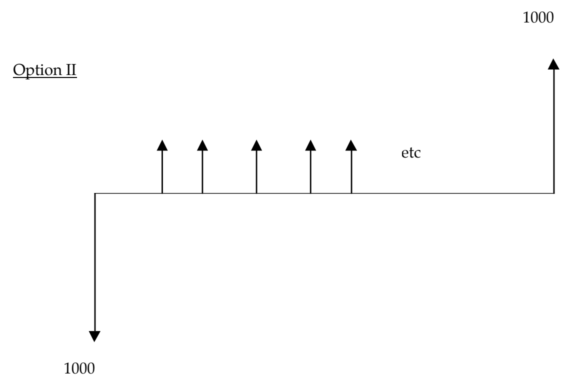
<u>Option I</u>

Return on investment in this case, i, is given by:

$$-513.6 + \frac{1000}{(1+i)^{10}} = 0$$

where i is the interest rate per half-year period

Homework problem (contd.)



Return on investment, j, in this case is given by

$$-1000 + 67(P/A, j\%, 10) + 1000(P/F, j\%, 10) = 0$$

Effect of inflation on bond value

- Inflation causes the purchasing power of money to decline
- Note the difference between the earning power and the purchasing power of funds
- See:
  - The effect of inflation on the value of bond income
- (Note -- same mistake on 'total present value' as before.)

Continuous compounding

- For the case of m compounding periods per year and nominal annual interest rate, r, the effective annual interest rate i_a is given by:

  $i_a = (1 + r/m)^m - 1$
- In the limiting case of continuous compounding

$$i_a = \lim_{m \to \infty} \left(1 + \frac{r}{m}\right)^m - 1$$

Writing

$$i = r/m$$

$$i_a = \lim_{i \to 0} (1 + i)^{\frac{r}{i}} - 1$$

$$= e^r - 1$$

or

$$r = \ln(1 + i_a)$$

Effective interest rates, i_a, for various nominal rates, r, and compounding frequencies, m

Compounding frequency	Compounding periods per year,m	Effective rate i_a for nominal rate of 6%	8%	10%	12%	15%	24%
Annually	1	6.00	8.00	10.00	12.00	15.00	24.00
Semiannually	2	6.09	8.16	10.25	12.36	15.56	25.44
Quarterly	4	6.14	8.24	10.38	12.55	15.87	26.25
Bimonthly	6	6.15	8.27	10.43	12.62	15.97	26.53
Monthly	12	6.17	8.30	10.47	12.68	16.08	26.82
Daily	365	6.18	8.33	10.52	12.75	16.18	27.11
Continuous	**∞**	**6.18**	**8.33**	**10.52**	**12.75**	**16.18**	**27.12**

Continuous Compounding, Discrete Cash Flows

(nominal annual interest rate r, continuously compounded, N periods)

To Find	Given	Factor Name	Factor Symbol	Factor formula
F	P	Future Worth Factor*	(F/P, r%, N)	F = $P(e^{rN})$
P	F	Present Worth Factor	(P/F, r%, N)	$P = F(e^{-rN})$
F	A	Future Worth of an annuity factor	(F/A, r%, N)	$F = A\left[\frac{e^{rN}-1}{e^{r}-1}\right]$
A	F	Sinking Fund Factor	(A/F, r%, N)	$A = P\left[\frac{e^{r}-1}{e^{rN}-1}\right]$
P	A	Present Worth of an annuity Factor	(P/A, r%, N)	$P = A\left[\frac{e^{rN}-1}{e^{rN}(e^{r}-1)}\right]$
A	P	Capital Recovery Factor	(A/P, r%, N)	$A = P\left[\frac{e^{rN}(e^{r}-1)}{e^{rN}-1}\right]$

Example:

You need $25,000 immediately in order to make a down payment on a new home. Suppose that you can borrow the money from your insurance company. You will be required to repay the loan in equal payments, made every 6 months over the next 8 years. The nominal interest rate being charged is 7% compounded continuously. What is the amount of each payment?

<u>Another example: Discrete compounding & discrete cash flows, but the compounding and payment periods don't coincide</u>

(From PSB:) Find the present worth of a series of quarterly payments of $1000 extending over 5 years, if the nominal interest rate is 8%, compounded monthly.

<u>Continuous cash flows, continuous compounding</u>

In many applications, cash flows are also essentially continuous (or it is convenient to treat them as such). We need to develop time value factors equivalent to those we have obtained for discrete cash flows.

Let $\bar{X}$ = continuous rate of flow of cash over a period (in units of , e.g., \$/yr)

<u>Assume</u> also: continuous compounding at a nominal rate of r%/yr

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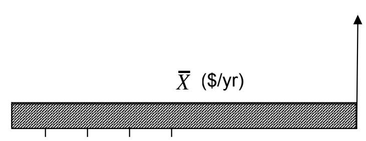
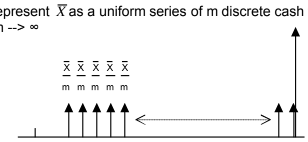


First, derive a future (end of year) worth X equivalent to 1 year of continuous cash flow $\bar{X}$ (\$/yr).

To do this, represent $\bar{X}$ as a uniform series of m discrete cash flows of magnitude $\bar{X}$/m in the limit as m --> ∞



Continuous cash flows, continuous compounding (contd.)

$$FW = \lim_{m \to \infty} \frac{\overline{X}}{m} \cdot (F/A,\ i\%,\ m) \quad \text{where} \quad i = \frac{r}{m}$$

$$X = \lim_{m \to \infty} \frac{\overline{X}}{m} \left[\frac{(1+i)^m - 1}{i} \right]$$

$$= \lim_{i \to 0} \overline{X} \left[\frac{(1+i)^{r/i} - 1}{r} \right]$$

$$= \overline{X} \left[\frac{e^r - 1}{r} \right] = \overline{X} \frac{i_a}{\ln(1 + i_a)}$$

Thus it is straightforward to convert a continuous cash flow into an equivalent series of uniform end-of-year discrete cash flows, and vice versa.

Example of continuous cash flows & continuous compounding

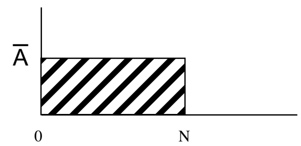
An oil refinery is considering an investment in upgrading a main pump. The upgrading is expected to result in a reduction in maintenance labor and materials costs of about $3000 per year.

If the expected lifetime of the pump is three years, what is the largest investment in the project that would be justified by the expected savings?

(Assume that the required rate of return on investments (before taxes) is a nominal rate of 20%/ year, continuously compounded.)

“Funds flow” time value factors for continuous cash flows and continuous compounding

- Instead of converting continuous cash flows to equivalent uniform discrete cash flows, you can alternatively use the ‘funds flow factors’ -- time value factors for continuous cash flows with continuous compounding



To Find:	Given:			
P	$\bar{A}$	$(P/\bar{A}, r\%, N)$	Present worth of a continuous annuity factor	$\frac{e^{rN} - 1}{re^{rN}}$
$\bar{A}$	P	$(\bar{A}/P, r\%, N)$	Continuous annuity from a present amount factor	$\frac{re^{rN}}{e^{rN} - 1}$
F	$\bar{P}$	$(F/\bar{P}, r\%, N)$	Future worth of a continuous present amount factor	$\frac{e^{rN}(e^{r} - 1)}{re^{r}}$
P	$\bar{F}$	$(P/\bar{F}, r\%, N)$	Present worth of a continuous future amount factor	$\frac{e^{r} - 1}{re^{rN}}$

Example (from PSB):

A county government is considering building a road from downtown to the airport to relieve congested traffic on the existing two-lane divided highway. Before allowing the sale of a bond to finance the road project, the county court has requested an estimate of future toll revenues over the bond life. The toll revenues are directly proportional to the growth of traffic over the years, so the following growth cash flow function is assumed to be reasonable:

$$F(t) = 5\,(1 - e^{-0.1t}) \qquad \text{(in millions of dollars)}$$

The bond is to be a 25-year instrument, and will pay interest at an annual rate of 6%, continuously compounded.