8.701

Introduction to Nuclear and Particle Physics

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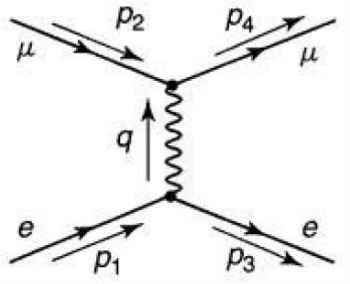
4. QED

4.8 Cross Sections

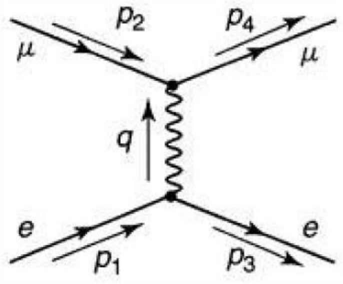
Cross Section Calculation



$$\begin{aligned}
\mathcal{M} &= -\frac{g_e^2}{(p_1-p_3)^2}\left[\bar{u}_3\gamma^\mu u_1\right]\left[\bar{u}_4\gamma_\mu u_2\right] \\
\left\langle|\mathcal{M}|^2\right\rangle &= \frac{g_e^4}{4(p_1-p_3)^4}\,\mathrm{Tr}\left[\gamma^\mu(\not{p}_1+m)\gamma^\nu(\not{p}_3+m)\right] \\
&\qquad\times\mathrm{Tr}\left[\gamma_\mu(\not{p}_2+M)\gamma_\nu(\not{p}_4+M)\right] \\
&= \frac{g_e^4}{4(p_1-p_3)^4}\left[4\left(p_1^\mu p_3^\nu+p_3^\mu p_1^\nu+(m^2-p_1\cdot p_3)g^{\mu\nu}\right)\right] \\
&\qquad\times\left[4\left(p_{2\mu}p_{4\nu}+p_{4\mu}p_{2\nu}+(M^2-p_2\cdot p_4)g_{\mu\nu}\right)\right] \\
&= \frac{4g_e^4}{(p_1-p_3)^4}\{2(p_1\cdot p_2)(p_3\cdot p_4)+2(p_1\cdot p_4)(p_2\cdot p_3) \\
&\qquad+2m^2(p_2\cdot p_4)+2M^2(p_1\cdot p_3) \\
&\qquad-4(p_1\cdot p_3)(p_2\cdot p_4) \\
&\qquad+4(m^2-p_1\cdot p_3)(M^2-p_2\cdot p_4)\}
\end{aligned}$$

$$\begin{aligned}
\left\langle|\mathcal{M}|^2\right\rangle &= \frac{8g_e^4}{(p_1-p_3)^4}\{(p_1\cdot p_2)(p_3\cdot p_4)+(p_1\cdot p_4)(p_2\cdot p_3) \\
&\qquad-m^2(p_2\cdot p_4)-M^2(p_1\cdot p_3)+2m^2M^2\}
\end{aligned}$$

Mott Scattering



Assuming $M \gg m, E, \mathbf{p}$ and scattering in the lab frame. Neglecting the muon recoil.

Differential cross section

$$\frac{d\sigma}{d\Omega} = \frac{\left\langle |\mathcal{M}|^2 \right\rangle}{(8\pi M)^2}$$

Mott Scattering - Kinematics

$$p_1 = (E, \mathbf{p}_1) \qquad p_2 = (M, \mathbf{0}) \qquad p_3 \simeq (E, \mathbf{p}_3) \qquad p_4 \simeq (M, \mathbf{0})$$

$$\begin{aligned}
(p_1 - p_3)^2 &= (0, \mathbf{p}_1 - \mathbf{p}_3)^2 \\
&= -\mathbf{p}_1^2 - \mathbf{p}_3^2 + 2\mathbf{p}_1 \cdot \mathbf{p}_3 \\
&= -2\mathbf{p}^2(1 - \cos\theta) \\
&= -4\mathbf{p}^2 \sin^2 \frac{\theta}{2}
\end{aligned}$$

$$\begin{aligned}
(p_1 \cdot p_3) &= \left[p_1^2 + p_3^2 - (p_1 - p_3)^2\right]/2 \\
&= m^2 + 2\mathbf{p}^2 \sin^2 \frac{\theta}{2} \\
(p_2 \cdot p_4) &= M^2 \\
(p_1 \cdot p_2) &= ME \\
(p_3 \cdot p_4) &= ME \\
(p_1 \cdot p_4) &= ME \\
(p_2 \cdot p_3) &= ME
\end{aligned}$$

Mott Scattering

$$\begin{aligned}
\left\langle |\mathcal{M}|^2 \right\rangle &= \frac{8g_e^4}{(p_1 - p_3)^4} \{(p_1 \cdot p_2)(p_3 \cdot p_4) + (p_1 \cdot p_4)(p_2 \cdot p_3) \\
&\qquad - m^2(p_2 \cdot p_4) - M^2(p_1 \cdot p_3) + 2m^2M^2\} \\
&= \frac{g_e^4}{2\mathbf{p}^4 \sin^4 \frac{\theta}{2}} \{2M^2E^2 - m^2M^2 \\
&\qquad - M^2(m^2 + 2\mathbf{p}^2 \sin^2(\theta/2)) + 2m^2M^2\} \\
&= \left(\frac{g_e^2 M}{\mathbf{p}^2 \sin^2(\theta/2)}\right)^2 \{E^2 - \mathbf{p}^2 \sin^2(\theta/2)\} \\
&= \left(\frac{g_e^2 M}{\mathbf{p}^2 \sin^2(\theta/2)}\right)^2 \{m^2 + \mathbf{p}^2 \cos^2(\theta/2)\}
\end{aligned}$$

Mott Scattering

$$
\begin{aligned}
\frac{d\sigma}{d\Omega} &= \left(\frac{1}{8\pi M}\right)^2 \left(\frac{g_e^2 M}{\mathbf{p}^2 \sin^2(\theta/2)}\right)^2 \left\{m^2 + \mathbf{p}^2 \cos^2(\theta/2)\right\} \\
&= \left(\frac{\alpha}{2\mathbf{p}^2 \sin^2(\theta/2)}\right)^2 \left\{m^2 + \mathbf{p}^2 \cos^2(\theta/2)\right\}
\end{aligned}
$$

This is the Mott formula. It describes Coulomb scattering off a nucleus, as long as the scattering particle is not too heavy or energetic. It assumes that the target is point-like.

Rutherford Scattering

If the initial-state particles are non-relativistic, the Mott formula simplifies further to the Rutherford scattering formula

$$\{m^2 + \mathbf{p}^2 \cos^2(\theta/2)\} \quad \rightarrow \quad m^2$$

$$\mathbf{p}^2 \quad \rightarrow \quad 2mE \qquad (E \text{ is kinetic energy})$$

$$\alpha \quad \rightarrow \quad q_1 q_2 \qquad \text{(Gaussian units)}$$

$$\frac{d\sigma}{d\Omega} \quad = \quad \left(\frac{q_1 q_2}{4E \sin^2(\theta/2)} \right)^2$$