8.701

Introduction to Nuclear and Particle Physics

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4. QED

4.2 Dirac Equation Solutions

Solutions

Looking for a free particle wave solution of form:

$$\psi(\mathbf{x}, t) = u(E, \mathbf{p})e^{i(\mathbf{p}\cdot\mathbf{x} - Et)}$$

Satisfying the Dirac equation

$$(i\gamma^\mu \partial_\mu - m)\psi = 0.$$

As Dirac spinor u(E,p) is a function of energy and momentum, derivatives only act on exponent.

$$\partial_0\psi \equiv \frac{\partial\psi}{\partial t} = -iE\psi, \quad \partial_1\psi \equiv \frac{\partial\psi}{\partial x} = ip_x\psi, \quad \partial_2\psi = ip_y\psi \quad \text{and} \quad \partial_3\psi = ip_z\psi$$

$$(\gamma^0 E - \gamma^1 p_x - \gamma^2 p_y - \gamma^3 p_z - m)u(E, \mathbf{p})e^{i(\mathbf{p}\cdot\mathbf{x} - Et)} = 0.$$

$$\boxed{(\gamma^\mu p_\mu - m)\, u = 0}$$

Particle at rest

p=0:

$$\psi = u(E,0)e^{-iEt}.$$

$$\boxed{(\gamma^\mu p_\mu - m)\,u = 0} \longrightarrow E\gamma^0 u = mu.$$

γ⁰ is diagonal

$$\psi_1 = N\begin{pmatrix}1\\0\\0\\0\end{pmatrix}e^{-imt},\ \psi_2 = N\begin{pmatrix}0\\1\\0\\0\end{pmatrix}e^{-imt},\ \psi_3 = N\begin{pmatrix}0\\0\\1\\0\end{pmatrix}e^{+imt}\ \text{and}\ \psi_4 = N\begin{pmatrix}0\\0\\0\\1\end{pmatrix}e^{+imt}.$$

General free particle

General solutions can be obtained by applying Lorentz transformation or directly from the Dirac equation for the spinor u(E,p)

$$(\gamma^\mu p_\mu - m)u = 0 \quad \rightarrow \quad (E\gamma^0 - p_x\gamma^1 - p_y\gamma^2 - p_z\gamma^3 - m)u = 0.$$

$$\left[\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} E - \begin{pmatrix} 0 & \boldsymbol{\sigma}\cdot\mathbf{p} \\ -\boldsymbol{\sigma}\cdot\mathbf{p} & 0 \end{pmatrix} - m\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}\right] u = 0.$$

$$\boldsymbol{\sigma}\cdot\mathbf{p} \equiv \sigma_x p_x + \sigma_x p_y + \sigma_x p_z = \begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{pmatrix}$$

General free particle

$$u = \begin{pmatrix} u_A \\ u_B \end{pmatrix} \rightarrow \begin{pmatrix} (E-m)I & -\boldsymbol{\sigma}\cdot\mathbf{p} \\ \boldsymbol{\sigma}\cdot\mathbf{p} & -(E+m)I \end{pmatrix}\begin{pmatrix} u_A \\ u_B \end{pmatrix} = 0, \rightarrow \begin{aligned} u_A &= \frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{E-m}u_B, \\ u_B &= \frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{E+m}u_A. \end{aligned}$$

Solutions are of the form $\psi_i = u_i(E,\mathbf{p})e^{i(\mathbf{p}\cdot\mathbf{x}-Et)}$

with

$$u_1 = N_1\begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \end{pmatrix}, \quad u_2 = N_2\begin{pmatrix} 0 \\ 1 \\ \frac{p_x-ip_y}{E+m} \\ \frac{-p_z}{E+m} \end{pmatrix}, \quad u_3 = N_3\begin{pmatrix} \frac{p_z}{E-m} \\ \frac{p_x+ip_y}{E-m} \\ 1 \\ 0 \end{pmatrix} \quad u_4 = N_4\begin{pmatrix} \frac{p_x-ip_y}{E-m} \\ \frac{-p_z}{E-m} \\ 0 \\ 1 \end{pmatrix}$$