22.101 Applied Nuclear Physics (Fall 2006)

Lecture 8 (10/4/06)

Neutron-Proton Scattering

References:

M. A. Preston, *Physics of the Nucleus* (Addison-Wesley, Reading, 1962).

E. Segre, *Nuclei and Particles* (W. A. Benjamin, New York, 1965), Chap. X.

We continue the study of the neutron-proton system by taking up the well-known problem of neutron scattering in hydrogen. The scattering cross section has been carefully measured to be 20.4 barns over a wide energy range. Our intent is to apply the method of phase shifts summarized in the preceding lecture to this problem. We see very quickly that the s-wave approximation (the condition of interaction at low energy) is very well justified in the neutron energy range of 1 - 1000 eV. The scattering-state solution, with E > 0, gives us the phase shift or equivalently the scattering length. This calculation yields a cross section of 2.3 barns which is considerably different from the experimental value. The reason for the discrepancy lies in the fact that we have not taken into account the spin-dependent nature of the n-p interaction. The neutron and proton spins can form two distinct spin configurations, the two spins being parallel (triplet state) or anti-parallel (singlet), each giving rise to a scattering length. When this is taken into account, the new estimate is quite close to the experimental value. The conclusion is therefore that n-p interaction is spin-dependent and that the anomalously large value of the hydrogen scattering cross section for neutrons is really due to this aspect of the nuclear force.

For the scattering problem our task is to solve the radial wave equation for s-wave for solutions with E > 0. The interior and exterior solutions have the form

$$u(r) = B\sin(K'r) \ , \qquad r < r_o \tag{8.1}$$

and

$$u(r) = C\sin(kr + \delta_o) \ , \quad r > r_o \tag{8.2}$$

where $K' = \sqrt{m(V_o + E)}/\hbar$ and $k = \sqrt{mE}/\hbar$. Applying the interface condition we obtain

$$K'\cot(K'r_o) = k\cot(kr_o + \delta_o) \tag{8.3)}$$

which is the relation that allows the phase shift to be determined in terms of the potential parameters and the incoming energy E. We can simplify the task of estimating the phase shift by recalling that the phase shift is simply related to the scattering length by $\delta_o = -ak$ (cf. (7.22)). Assuming the scattering length a is larger than r_o, we see the RHS of (8.3) is approximately $k\cot(\delta_o)$. For the LHS, we will ignore E relative to V_o in K', and at the same time ignore E_B relative to V_o in K. Then K' ~ K and the LHS can be set equal to $-\kappa$ by virtue of (6.4). Notice that this series of approximations has enabled us to make use of the dispersion relation in the bound-state problem, (6.4), for the scattering calculation. As a result, (8.3) becomes

$$k\cot(\delta_o) = -\kappa \tag{8.4}$$

which is a relation between the phase shift and the binding energy.

Once the phase shift δ_o is known, the differential scattering cross section is then given by (7.20),

$$\sigma(\theta) = (1/k^2)\sin^2\delta_o \tag{8.5}$$

A simple way to make use of (8.4) is to note the trigonometric relation $\sin^2 x = 1/(1+\cot^2 x)$, or

$$\sin^2\delta_o = \frac{1}{1+\cot^2\delta_o} = \frac{1}{1+\kappa^2/k^2} \tag{8.6}$$

Thus,

$$\sigma(\theta) \approx \frac{1}{k^2 + \kappa^2} = \frac{\hbar^2}{m} \frac{1}{E + E_B} \approx \frac{\hbar^2}{m E_B} \tag{8.7}$$

The last step follows because we are mostly interested in estimating the scattering cross section in the energy range 1 - 100 eV. Putting in the numerical values of the constants, $\hbar = 1.055$ x 10^{-27} erg sec, m = 1.67 x 10^{-24} g, and and E_B = 2.23 x 10^6 x 1.6 x 10^{-12} ergs, we get

$$\sigma = 4\pi\hbar^2 / mE_B \sim 2.3 \text{ barns} \tag{8.8}$$

This value is considerably lower than the experimental value of the scattering cross section of H^1, 20.4 barns, as shown in Fig. 8.1.

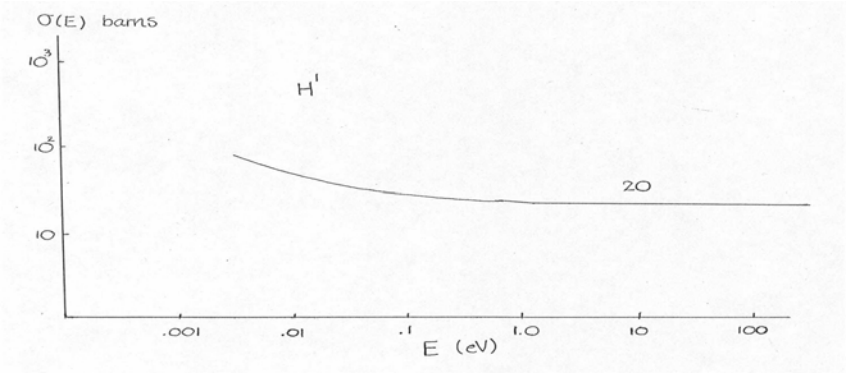

Fig. 8.1. Experimental neutron scattering cross section of hydrogen, showing a constant value of 20.4 barns over a wide range of neutron energy. The rise in the cross section at energies below ~ 0.1 eV can be explained in terms of chemical binding effects in the scattering sample.

The explanation of this well-known discrepancy lies in the neglect of spin-dependent effects. It was suggested by E. P. Wigner in 1933 that neutron-proton scattering should depend on whether the neutron and proton spins are oriented in a parallel configuration (the triplet state, total spin angular momentum equal to $\hbar$) or in an anti-parallel configuration (singlet state, total spin is zero). In each case the interaction

potential is different, and therefore the phase shifts also would be different. Following this idea, one can write instead of (8.7),

$$\sigma(\theta) = \frac{1}{k^2}\left(\frac{1}{4}\sin^2\delta_{os} + \frac{3}{4}\sin^2\delta_{ot}\right) \tag{8.9}$$

We have already mentioned that the ground state of the deuteron is a triplet state at E = -E_B. If the singlet state produces a virtual state of energy E = E*, then (8.9) would become

$$\sigma \approx \frac{\pi\hbar^2}{m}\left(\frac{3}{E_B} + \frac{1}{E^*}\right) \tag{8.10}$$

Taking a value of E* ~ 70 keV, we find from (8.10) a value of 20.4 barns, thus bringing the theory into agreement with experiment.

In summary, experimental measurements have given the following scattering lengths for the two types of n-p interactions, triplet and singlet configurations, and their corresponding potential range and well depth.

Interaction	Scattering length a [F]	r_o [F]	V_o [MeV]
n-p (triplet)	5.4	2	36
n-p (singlet)	-23.7	~ 2.5	18

Notice that the scattering length for the triplet state is positive, while that for the singlet state is negative. This illustrates the point of Fig. 7.3.

As a final remark, we note that experiments have shown that the total angular momentum (nuclear spin) of the deuteron ground state is I = 1, where $\underline{I} = \underline{L} + \underline{S}$, with $\underline{L}$ being the orbital angular momentum, and $\underline{S}$ the intrinsic spin, $\underline{S} = \underline{s}_n + \underline{s}_p$. It is also known that the ground state is mostly 1s (ℓ =0), therefore for this state we have S = 1

(neutron and proton spins are parallel). We have seen from Lec 6 that the deuteron ground state is barely bound at E_B = 2.23 Mev, so all the higher energy states are not bound. The 1s state with S = 0 (neutron and proton spins antiparallel), is a virtual state; it is unbound by ~ 60 Kev. An important implication is that nuclear interaction varies with S, or, *nuclear forces are spin-dependent*.

Effects of Pauli Exclusion Principle

One might ask why are the di-neutron and the di-proton unstable. The answer lies in the indistinguishability of particles and the Exclusion Principle (no two fermions can occupy the same state). Consider the two electrons in a helium atom. Their wave function may be written as

$$\psi(1,2) = \psi_1(\underline{r}_1)\psi(\underline{r}_2)$$

$$= A\frac{\sin k_1 r_1}{r_1}\frac{\sin k_2 r_2}{r_2} \qquad (8.11)$$

where $\psi_1(\underline{r})$ is the wave functrion of electron 1 at $\underline{r}$. But since we cannot distinguish between electrons 1 and 2, we must get the same probability of finding these electrons if we exchange their positions (or exchange the particles),

$$|\psi(1,2)|^2 = |\psi(2,1)|^2 \quad \Rightarrow \quad \psi(1,2) = \pm\psi(2,1)$$

For fermions (electrons, neutrons, protons) we must choose the (-) sign; because of Fermi-Dirac statistics the wave function must be anti-symmetric under exchange. Thus we should modify (8.11) and write

$$\psi(1,2) = \psi_1(\underline{r}_1)\psi_2(\underline{r}_2) \; - \; \psi_2(\underline{r}_1)\psi_1(\underline{r}_2) \; \equiv \; \psi_-$$

$$+ \qquad\qquad \psi_+$$

If we now include the spin, then an acceptable anti-symmetric wave function is

$$\Psi(1,2) = \psi_{-}\chi_1(\uparrow)\chi_2(\uparrow)$$

so that under an interchange of particles, $1 \leftrightarrow 2$, $\psi(1,2) = -\psi(2,1)$. This corresponds to S = 1, symmetric state in spin space. But another acceptable anti-symmetric wave function is

$$\Psi_{-}(1,2) = \psi_{+}[\chi_1(\uparrow)\chi_2(\downarrow) - \chi_1(\downarrow)\chi_2(\uparrow)]$$

which corresponds to S = 0, anti-symmetric state in spin.

For the symmetry of the wave function in configurational space we recall that we have

$$\psi(\underline{r}) \sim \frac{u_\ell(r)}{r} P_\ell^m(\cos\theta) e^{im\varphi}$$

which is even (odd) if ℓ is even (odd). Thus, since Ψ has to be anti-symmetric, one can have two possibilities,

ℓ even, S = 0 (space symmetric, spin anti-symmetric)

ℓ odd, S = 1 (space anti-symmetric, spin symmetric)

These are called T = 1 states (T is isobaric spin), available to the n-p, n-n, p-p systems. By contrast, states which are symmetric (T = 0) are

ℓ even, S = 1

ℓ odd, S = 0

These are available only to the n-p system for which there is no Pauli Exclusion Principle.

The ground state of the deuteron is therefore a T = 0 state. The lowest T = 1 state is ℓ = 0, S =0. As mentioned above, this is known to be unbound (E ~ 60 Kev). We

should therefore expect that the lowest T = 1 state in n-n and p-p to be also unbound, i.e., there is no stable di-neutron or di-proton.

Essential Features of Nuclear Forces

In closing we summarize a number of important features of the nucleon-nucleon interaction potential, several of which are basic to the studies in this class [Meyerhof, Chap. 6].

1. There is a dominant short-range part, which is central and which provides the overall shell-model potential.
2. There is a part whose range is much smaller than the nuclear radius, which tends to make the nucleus spherical and to pair up nucleons.
3. There is a part whose range is of the order of the nuclear radius, which tends to distort the nucleus.
4. There is a spin-orbit interaction.
5. There is a spin-spin interaction.
6. The force is charge independent (Coulomb interaction excluded).
7. The force saturates.